MSU BUSINESS STUDIES

Elementary Mathematics of
LINEAR PROGRAMMING AND GAME THEORY

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To
Wallace E. Pratt
and
Jay E. Crane

FOREWORD

The most striking breakthrough in business and economic research in the past two decades has been the development of new empirical techniques which are frequently classified under the somewhat nebulous title of "operations research." Perhaps the most fruitful of these new techniques is linear programming, an outgrowth of game theory concepts which were first conceived by the late John von Neumann in the 1920's.

While the pragmatic value of mathematical programming is incontestable, the extent of its contribution to the content of economic theory may remain unsettled for many years. Still, it seems clear that an understanding of the rationale of linear programming and game theory is mandatory for both the practicing business economist and the economic theorist. A majority of the practicing business economists and many academic economists may be inadequately prepared mathematically, however, to learn what they need to know about these subjects.

In view of this state of affairs it is easy to understand why I welcome a book which breaks this impasse by presenting an accurate and largely self-contained exposition, bringing these subjects within the grasp of intelligent people with little training in higher mathematics.

While a knowledge of college algebra is essential and an acquaintance with analytical geometry desirable, the most important requirements for studying Mr. Bennion's amazingly clear and patiently developed exposition are sufficient motivation and staying power. These qualities are standard equipment with many business economists and most academicians.

Edward G. Bennion is Consulting Economist for the General Economics Department of the Standard Oil Company (New Jersey). He received his Ph.D. from Harvard, and has taught there as well as at the Massachusetts Institute of Technology, Yale and the Columbia Graduate School of Business. The general economic work at Jersey was conceived and developed under his leadership and headed by him from its inception in 1945 until 1953, when he abandoned administration for his present activities. One of his major interests in the research field has been the development and application of methods to improve the value of general economic work in business decision-making. The present book is the outgrowth of that interest.

Richard C. Henshaw, Jr.
Professor of Business Administration
Michigan State University

PREFACE

ALEXANDER GRAY once expressed the view that the outrageous imbalance between the supply of books and the demand for them made it incumbent on the would-be author to advise his potential audience as to just what burning need he imagined his own literary gem might fill.[1] Some attempt in this direction is made, for this volume, in the early pages of the first chapter. This makes it possible to devote the preface to warning the reader of the author's limited qualifications, and to making the customary acknowledgments, which are largely in the form of apologies for such flagrant plagiarism.

This book was written for what might be called the curious-mathematical-lowbrow class, though the thoughtful reader may elevate himself to the low-middle brow group before he is through. The author is a charter member of the curious-lowbrow fraternity, and has made a conscientious attempt to educate himself on what seem to him the more important mathematical fundamentals. These are his principal qualifications for writing a book on this subject, since one who has gone through the ordeal of self-education in any mathematical area is almost sure to have a greater awareness of the problems of this learning process than would a man with more formal training.

Having learned the hard way, the writer has tried to initiate the coverage of this book at a reasonably low mathematical level. At least to the writer's knowledge, there is no other book available which has the objective of imparting an understanding of the rationale underlying linear programming and game theory and which also assumes so little advanced mathematical knowledge at the outset. And, as a further aid to his fellow lowbrows, the writer has generally chosen the somewhat unscientific course of backing into mathematical generalizations from specific examples.

Following this course makes easier, for the mathematically unsophisticated, an understanding of mathematical generalizations. None of this means, however, that one can expect simple answers to complicated problems, and it would be misleading to pretend that this is a simple book from other than a mathematician's viewpoint. The ultimate generalizations are vital to an understanding of rationale; and these generalizations sometimes necessarily rest on rather abstract chains of reasoning involving compact mathematical notation. Neither the reasoning nor the notation are immediately second-nature to the non-mathematician, even when he is eased into these as gently as the writer has tried to do.

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[1] Alexander Gray, *The Development of Economic Doctrine* (New York: Longmans, Green and Company, 1933), p. 5.

This compromising approach unquestionably leaves more to intuition than is desirable from a mathematician's viewpoint, and the writer would be among the first to defend the mathematician's viewpoint. The most desirable method of learning would clearly be for each of us to sit at the feet of bona fide mathematicians; and such is rightly one of the current objectives in secondary schools and institutions of higher education. But this is too expensive and slow for those of us who lacked the foresight to grasp this opportunity in our youth. This no-longer-so-youthful group has, if the area covered by this book is an important one, only two alternatives: to roll up its sleeves and sweat a bit in the interests of self-education, or to risk withering slowly on the vine. This book was written to help those who wish to choose the former course.

While most of those to whom the author is indebted for this volume are known only to himself, it is desirable to make some explicit acknowledgments—in the interest of a clear conscience, if for no other reason. Directly or indirectly, and almost by definition, most books savoring of a text might be regarded as a sort of ethically acceptable form of plagiarism. This book is a blatant example of this type of theft, since its contribution (if it may be dignified with that description) is to bring its subject matter more easily within the reach of the mathematically unsophisticated. The least the writer can do, therefore, is to indicate those he has drawn upon most.

The coverage, approach, and organization of this book—along with such vital underlying understanding as the author hopes he ultimately came to possess—have been largely drawn from three sources: (1) Charnes, Cooper, and Henderson, *An Introduction to Linear Programming*, Part II (John Wiley and Sons, Inc., 1953); (2) Dorfman, Samuelson, and Solow, *Linear Programming and Economic Analysis* (McGraw-Hill Book Company, Inc., 1958); and (3) Vajda, *The Theory of Games and Linear Programming* (Methuen and Co., Ltd., 1956). There are numerous other fine books in this field on the market, of course, but none with which the writer is familiar comes quite as close to meeting this writer's requirements—and even these three require rather substantial effort and more mathematical sophistication than one normally imputes to mathematical lowbrows.

The writer has been guilty of no more than petty larceny with respect to the first of these sources, but he has been guilty of something closer to grand larceny with respect to the second source for the concluding chapter of this book. This stoop to levels no lower than petty larceny and limited grand larceny should not be regarded as any less of a compliment. Even if the market abounded (which it does not) with books relating the techniques of game theory and linear programming to economic analysis, for example, it is hard to imagine a better, or to economists a more imperative, book than Dorfman, Samuelson, and Solow, which is written with the clarity one always expects from these authors. But one must first learn to walk among the techniques themselves, before attempting to run in a particular field in which they can be applied. This volume has, for the most part, been intentionally confined to the simpler objective of merely learning to walk.

Even if it is not entirely obvious to the reader, considerable grand larceny has been employed in the writer's drawing upon Vajda's fine little book, primarily because Vajda's coverage and (in somewhat lesser degree) his organization agree so closely with the writer's notion of what should be the first step in an induction into the techniques of linear programming and game theory. Vajda's book is more than a book simply having a coverage and organization with which the writer is in sympathy; it is also a quite well-written book. Its real drawback is that it is not self-contained, except for one with a fair degree of mathematical maturity.

Unless one understands the rudiments of matrix algebra notation, has a passably broad knowledge of matrix algebra concepts, and is sufficiently facile in mathematics to be able readily to fill in steps in proofs that are omitted (presumably because these steps are too obvious to the sophisticated to warrant wasting space on them)—unless one can meet these three requirements, one is sure to have a difficult time achieving a real understanding of the fundamentals of linear programming and game theory from Vajda's little book. On the other hand, for one who can satisfy these requirements, Vajda's book is, in the writer's opinion, to those earnestly seeking to learn to walk with some assurance among the techniques involved, what Dorfman, Samuelson, and Solow is to the properly grounded economist.

While the writer profited substantially from numerous additional works in this field and would recommend these works with equal enthusiasm for the person who has passed the initial growing-pains stage, his real debt with respect to this volume is to the three abovementioned books. To learn a little about the what, and even about the how of linear programming and game theory, is a simple (though decidedly unsatisfying) chore. To couple these with the much more fundamental why of linear programming and game theory techniques is, however, a truly rugged and time-consuming ordeal for the mathematically unsophisticated. The task would have been close to impossible, had not those three books been available to the writer. It is the writer's hope that this book will further reduce the ruggedness of the ordeal without materially sacrificing anything essential to fundamental understanding.

These prefatory remarks would be less than gracious were specific tribute not paid to several other people, all of whom have left their imprint upon this book.

Professor Richard C. Henshaw, Jr. must, for better or worse, bear much of the responsibility for the writer's reflections appearing in book form. It was Professor Henshaw who first exposed a group of students to these reflections, by requesting copies of the original manuscript to be used as a textbook in a course of his at Michigan State University, early in 1959. As a result of this experience, Professor Henshaw was the first person to raise the question of publication, and the writer has further benefited substantially from Professor Henshaw's other suggestions.

The writer is also deeply indebted to his colleague, Mr. C. Ashley Wright, and to Professor Harvey M. Wagner of Stanford University. Both Mr. Wright and Professor Wagner were generous enough to read the entire manuscript, in an effort to protect the writer from his own ignorance. It should be added, especially since the writer has not always followed the excellent advice of Mr. Wright and Professors Henshaw and Wagner, that the fault for any remaining manifestations of ignorance must be laid solely at the writer's door.

The writer's indebtedness to his secretary, Miss Evelyn T. Benecke, is no less considerable. Miss Benecke nurtured the writer's dreadful scrawl through the several stages necessary to bring it to printed form with meticulous care and zealous interest. Unquestionably, Miss Benecke's devotion to the job both greatly reduced the chances of typographical error and saved the writer many hours of thankless work.

Finally, to lapse briefly into the first person singular, I have dedicated this book to two men widely known and highly respected in their own fields of finance and geology: Messrs. Jay E. Crane and Wallace E. Pratt, both former Directors and Vice Presidents of Standard Oil Company (New Jersey). Over the past fifteen years, these two men have wielded a far greater influence (invariably for the better) on my activities and thinking than they probably realize; and, since both are now retired, the bounds of propriety do not preclude my paying them this small tribute. Few people I have known can rival these two men either in my affection for them as persons or in my respect for their wisdom.

E. G. BENNION

Riverside, Connecticut
February, 1960

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CHAPTER I

Prologue: Declamation and Orientation

Linear programming and game theory are important subjects. While this has been true since their conception in an academic atmosphere, their more recent use outside the academic world constitutes a substantially widened recognition of the importance of these. To the academician, linear programming and game theory are viewed less as decision-making aids than as analytical techniques, while the converse is true for the businessman or the war strategist; but the importance is undeniable in either case.

The spread of recognition has not, however, been anything like synonymous with a spread of understanding of the fundamentals of linear programming and game theory. Even in the academic world, and to a much greater extent outside of it, the spread of understanding has been confined to a relatively small group of experts. This is unfortunate, in the sense that it greatly limits the use of these valuable techniques in both worlds.

Unfortunate though the relatively limited spread of understanding may be, it is thoroughly understandable. For however important linear programming and game theory may be, acquiring an understanding of them is undeniably an onerous task for the mathematically unsophisticated. This is certainly not because of a scarcity of literature on these subjects.

In addition to the trail-blazing works in the fields of linear programming and game theory, there is now available a considerable number of genuinely excellent books that are more expository in nature. Each of these achieves its objectives, and all of these objectives are desirable. These objectives are such, however, that either a knowledge of the calculus or of matrix algebra (or both) on the part of the reader is assumed, or the coverage is so thorough as to lead to a rather difficult book of quite substantial length. Either of these courses makes the acquisition of an understanding of the fundamentals of linear programming and game theory, by a mathematically unsophisticated reader, if not an impossible task, certainly an extremely demanding one.

OBJECTIVE AND CONTENT

This book has the decidedly limited objective of making it possible for the reader with more intellectual curiosity than advanced mathematical training to grasp the fundamentals, or rationale, underlying linear programming and game theory. Its writing is the result of the attempt of a mathematical illiterate to teach himself, and a handful of others, something of this rationale. If, as is hoped, the result is an interpretation essentially devoid of errors, a mathematically-minded general reader can learn everything the author knows with a small fraction of the effort he himself expended in acquiring it.

Even this elementary book cannot be read without thought, and it tends to become increasingly complex as it unfolds. The book is, however, self-contained: the mathematically unsophisticated reader should be able to understand it, without recourse to mathematical textbooks (advanced, or otherwise), if he is willing to read with care and concentration.

It necessarily follows that a small book, concerned with only the major fundamentals, leaves a great deal unsaid. To cover what the writer sees as the major fundamentals is not to cover all the important ones, even if the writer's view is correct. Furthermore, a book which concerns itself with fundamentals must, if it is to remain both small and reasonably comprehensible to the mathematically unsophisticated, confine itself largely to the abstract, i.e., to small, invented problems, rather than to the generally unwieldy ones of real life. In this respect, the book reads like a high school textbook: the important generalizations are eventually drawn, but only after considering small problems using specific coefficients attached to abstract symbols.

The following section of this chapter does pose, for orientation purposes, a realistic problem, rather than one concerned with nameless, abstract symbols; and from time to time in subsequent chapters this same problem is developed further for illustrative purposes, as well as to remind the reader that abstract symbols can represent highly concrete things. But one realistic problem may be worse than none, if it gives the reader the impression that only a single-plant (in the terminology of the economist) problem of a particular type can be solved. Any problem (whether it be one of business, economics, war, or whatnot), which can be formulated in the same general type of frameworks as those considered under linear programming and game theory, is a problem to the solution of which these techniques may be able to make a substantial contribution. For this reason, among others, a genuine understanding of these techniques in the abstract should be the reader's real objective. The justification for an initial realistic problem is simply to make the transition to the abstract relatively painless.

One final comment on the content of this book is in order. Short though the book is, the reader can shorten it yet more without seriously impairing either his ability to understand the remainder of the book or his understanding of the primary rationale underlying linear programming and game theory. There is much to be said for the inclusion of Chapters IV and V in a book of this sort. Nevertheless, the reader *can* dispense with these two chapters with but little loss to his understanding of rationale. But this carte blanche cannot be safely extended to the reader who expects to attempt a solution to problems: sooner or later he will tangle with a problem having no finite optimum, or with a problem of potential degeneracy. Without Chapters IV and V under his belt, he will then begin to wonder what has gone wrong with his hard-won knowledge.

Having briefly considered the importance of linear programming and game theory, the principal objective of this book, and the general nature of the book's content, the reader should now be sufficiently interested (and softened up) to plunge headlong into linear programming. Before making this headlong plunge, however, it might be helpful to acclimatize ourselves a bit by first dabbling our toes in the waters of an orientation problem.

AN ORIENTATION PROBLEM

For the mathematically unsophisticated, linear programming is more readily understandable if the mathematical generalizations are preceded by relatively simple examples. Unfortunately, there are few real examples in as simple dimensions as are desirable for this purpose. To remedy this lack we can fall back upon small problems with specific numbers, but these problems will still necessarily involve abstract symbols and not be so obviously related to real problems. For this reason, one is likely to find the trees obscuring one's view of the forest, i.e., the relation of all the details to the end purpose may not be easily discernible.

It seems sensible, therefore, to begin our inquiry into linear programming by considering a realistic problem. If we examine the nature of such a problem, along with a few of the steps involved in solving it, and if we learn the solution to that problem, even though we do not yet fully understand how that solution is reached, consideration of little examples not specifically related to some tangible problem then becomes more meaningful.

Consider the following concrete problem.[1] A refinery produces, as a by-product, 1,000 gallons per hour of virgin pitch in its crude distillation operation. Commercial fuel oil, having a realization (or sales) value of 5 cents per gallon, can be derived from the pitch in one of two ways:

(1) The pitch can be blended directly with flux stock, which has a cracking value (and hence represents a cost) of 8 cents per gallon, to make commercial fuel oil. The flux stock thus constitutes a cost of producing the commercial fuel oil, but we arbitrarily assign a zero cost value to our by-product, the pitch.

(2) The pitch can be blended indirectly with flux stock, either wholly or in part, by being first sent to a visbreaker unit capable of breaking the pitch (or visbreaker feed) down into an 80 per cent yield of tar, which tar can then be blended with flux stock to make commercial fuel oil. The cost of this additional operation, in conjunction with the different viscosity and gravity blend numbers of the resultant tar,[2] is such that we can assign a zero value to the tar, despite the fact that the quantity of tar derived is only 80 per cent as great as the quantity of pitch. The viscosity and gravity blend numbers of the pitch, tar, and flux are invariant at the figures shown in the table below. The viscosity and gravity blend numbers of the fuel oil can be varied by varying the proportion of flux stock used in the blending process, but we initially specify *exact* vis-

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[1] This case is taken from Chapter 2 of Gifford H. Symonds, *Linear Programming: The Solution of Refinery Problems* (New York: Esso Standard Oil Company, 1955).

[2] These numbers are the result of a technical definition (which need not concern us) intended to make possible the measuring of the viscous and weight properties of petroleum products in such a way that both the viscosity and the gravity blend numbers of (say) two products which are blended can be averaged to yield both the viscosity and the gravity blend numbers of the product resulting from the blending process. These numbers vary with varying viscosity and gravity in the opposite direction from that which one would think most logical, i.e., the *higher* the viscosity and gravity blend numbers, the *less* viscous and the *lighter* is the product—thus commercial fuel oil both pours more freely than, and is lighter than, tar; but both the viscosity and the blend numbers of commercial fuel oil are higher than those of tar.

cosity and gravity blend numbers for the commercial fuel oil of 21 and 12, respectively.

	Quantity Available (Gallons/Hour)	*Value (Cents/Gallon)*	*Viscosity Blend Number*	*Gravity Blend Number*
Pitch	$P = 1{,}000 - V$	0	5	8
Visbreaker Feed	V	0		
Tar	$T = 0.8V$	0	11	7
Flux	$F =$ any amount	8	37	24
Fuel Oil	$P + T + F$	5	21 (minimum)	12 (minimum)

The complete information at our disposal is listed in the table above. We pose as our initial problem that of determining the most profitable mixture of pitch, tar, and flux stock to obtain commercial fuel oil which just meets the exact viscosity and gravity specifications.

The information in the above table provides us with four requirements equations (requirements which must be met in the form of equations)—a viscosity requirement equation, a gravity requirement equation, and two material balance requirement equations:

Viscosity (3rd column): $5P + 11T + 37F = 21(P + T + F)$

Gravity (4th column): $8P + 7T + 24F = 12(P + T + F)$

Material (1st row and column): $P + V = 1{,}000$

(3rd row and 1st column): $T = 0.8V$

To understand the derivation of these equations, consider the viscosity requirement equation of $5P+11T+37F=21\ (P+T+F)$. In words, what this equation says is that we are required to combine as yet unknown quantities of pitch, tar, and flux (having viscosity blend numbers, respectively, of 5, 11, and 37) in such a way that the resulting blend (the $P+T+F$ on the right side of the equation) has a precise blend number of 21. The gravity requirement equation carries an analogous message. The third equation says we have 1,000 gallons of pitch, all of which we must use in as yet unknown proportions as a direct blend (P) or as an indirect blend (V) by sending it to the visbreaker unit to break it down into tar. The fourth equation merely states that, whatever amount (V) of pitch we do send to the visbreaker unit, the tar yield (T) will only be 80 per cent of that amount.

The coefficients of the above four equations can be depicted in what is sometimes called the *requirements space*, which is here four-dimensional (one dimension beyond our three-dimensional visualization capacities). And this requirements space is four-dimensional, *not* because we have the four unknowns of P, T, F, and V, *but because we have four constraints, or equations.* More specifically, as will be clearer from the geometric presentation of Chapter II, each set of *coefficients* associated with a *particular* unknown variable can be represented graphically as a single point in space. For example, the coefficients associated with F in the three equations on page 9 of this chapter are $(-2, -3, 0)$; and these coefficients are depicted as the single point, P_3, in Diagram 5 of Chapter II. Thus one needs, for such a graphic presentation, a separate axis for each of the coefficients associated with the particular unknown variable; and the number of these coefficients will necessarily be the same as the number of requirements equations (hence the term "requirements space").

Similarly, if we wished to do so, we could also depict the values of the *variables* yielded by a solution to the problem as a single point in what is sometimes known as the *solutions space.* In this case, however, we should need a separate axis for *each* of the unknown *variables* (rather than for *each* of the *coefficients* associated with a *particular* variable). Thus the solutions space would be two-dimensional for a problem involving two variables, three-dimensional for one involving three variables, and so on (hence the term "solutions space"). As a rule, and for reasons that will soon become obvious, linear programming problems run to a form in which the number of variables is greater than the number of requirements equations: i.e., the solutions space is of higher dimension than the requirements space.

Our above table of information can also provide us with a fifth equation. If our objective is to maximize profits (call them Z), we can say that "profits" will be:[3]

$$Z = 5(P + T + F) - 8F \text{(in cents per hour)}$$

This equation says the obvious. To determine our "profits," we must first find the sales value of our commercial fuel oil resulting from an appropriate blending of pitch, tar, and flux by multiplying the total number of gallons in that blend $(P+T+F)$ by the 5 cents per gallon sales value of commercial fuel oil. From that figure for total sales value, we must then subtract our only "cost"—namely, the number of gallons of flux (F) used in the blend multiplied by the 8 cents per gallon cost of flux. This will give us our profits (Z) in cents *per hour*, since the entire problem hinges on the fact that the refinery produces 1,000 gallons of virgin pitch *per hour*.

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[3] These are not "profits" in the economists' sense, since our only cost element is that of flux stock. Subject to the validity of our above assumptions, however, the "profits" functional will tell us the comparative "profits" figures (before all other costs, which are assumed invariant) from varying the mixture of pitch, tar, and flux stock.

This last equation above is known as the *functional*, since what we seek to maximize (in this case, profits) is a function of some (or all) of the unknowns residing in our solutions space. In other words, a linear programming problem is one of solving m requirements equations in n unknowns in such a way as to maximize (or minimize) a functional involving some (or all) of the n unknowns.

Our particular problem happens to have four requirements equations involving exactly four unknowns, (so it is a problem in which $m=n$, rather than $m<n$), and it has a unique solution.[4] This solution is:

$$F = 333$$

$$P = -333$$

$$V = 1{,}333$$

$$T = 1{,}067$$

In other words, these four values, *and these four values alone*, will satisfy the four requirements equations.

This clearly constitutes no solution to the problem we posed. For it is certainly beyond our capabilities to turn pitch into a negative quantity. Thus we may also say that solving a linear programming problem involves solving m equations in n unknowns, not only in such a way as to maximize the functional, but also in such a way as to have no negative unknowns. This can be done, but it requires a little contriving.

In essence, this contriving involves turning our first three requirements equations into inequalities, and then introducing an additional unknown into each inequality to return the inequality to the form of an equation. This procedure sounds a bit abstruse, so let us consider what it actually involves in this case.

By making the right sides of our viscosity and gravity equations equal to the left sides of these two equations, we were setting the problem up in such a way as to get an answer yielding the quantities of pitch, tar, and flux required to meet certain *exact* viscosity and gravity specifications of the product. If meeting these exact specifications were our objective, and if we substituted $0.8V$ for T into the viscosity and gravity equations, we could reduce those two equations to the following form:

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[4] For those who may have forgotten, an example of solving two simultaneous equations can be found in the second paragraph of the Matrix Algebra Section of Chapter III.

$$2P + V - 2F = 0^*$$
$$P + V - 3F = 0$$

Now, if we were willing to let the viscosity and gravity specifications *exceed* the exact specifications initially set, the left side of each of these last two equations might be less than zero by some (as yet unknown) amount—say y_1 for the viscosity equation and y_2 for the gravity equation. To admit the possibility, therefore, of such excess over the initial exact specifications, we should have to rephrase these last two equations to:

$$\begin{aligned} 2P + V - 2F + y_1 \quad\quad &= 0 \\ P + V - 3F \quad\quad + y_2 &= 0 \end{aligned}$$

To illustrate again with reference to the above viscosity requirement equation, this equation derives from our saying that, if we are willing to *exceed* the initial specification of an *exact* viscosity of 21 for commercial fuel oil, then $2P+V-2F \leq 0$. If the left side of this inequation does turn out to be <0, rather than $=0$, we need to know by how much it is less than zero, to determine the amount by which we must exceed the viscosity of 21 initially set. This as yet unknown amount we label as y_1, and y_1 must necessarily be either positive or zero. Consequently, the inequation of $2P+V-2F \leq 0$ can be stated in the form of an equation of $2P+V-2F+y_1=0$, where $y_1 \geq 0$.

By the same token, there is nothing sacred about our making $P+V=1{,}000$. If we can maximize profits better by making $P+V < 1{,}000$ (i.e., by throwing away part of the 1,000 gallons), we are certainly free to do so. Consequently, our first material-balance-re-

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[*] Consider, for example, the initial viscosity requirement equation:
(a) $5P+11T+37F=21(P+T+F)$.
Transposing the left side to the right side, we get:
(b) $16P+10T-16F=0$.
Substituting $0.8V$ for T, we have:
(c) $16P+8V-16F=0$.
And dividing through by 8:
(d) $2P+V-2F=0$.
Thus, equation (d) is the equivalent of equation (a).

quirement equation now takes the form of:

$$P + V + y_3 = 1{,}000,$$

where y_3 is the (as yet unknown) amount of the 1,000 gallons we throw away. Consequently, we emerge with a system of three equations in six unknowns:

$$\begin{aligned} 2P + V - 2F + y_1 \qquad\qquad &= 0 \\ P + V - 3F \qquad + y_2 \qquad &= 0 \\ P + V \qquad\qquad\qquad + y_3 &= 1{,}000 \end{aligned}$$

while our functional for maximizing profits remains the same as before:

$$Z = 5P + 4V - 3F$$

Our new problem has thus become one of finding non-negative values for the six variables P, V, F, y_1, y_2, and y_3 which will satisfy the three requirements equations in such a way as to maximize our functional. If there is such a non-negative solution to these three equations, it is highly probable that there is an *infinite number of non-negative solutions* (for a unique solution to a system of equations rarely exists, except when the number of unknowns exactly equals the number of equations), and ours is the task of finding, in that infinite number of solutions, the particular solution(s) which will maximize the profit function.

Such is the general nature of the linear programming problem. And, in our specific example, the optimal solution is actually:

$$P = 0, \quad y_1 = 0; \quad y_3 = 0; \quad V = 1{,}000; \quad F = 500; \quad y_2 = 500$$

In words, this solution tells us this:

(1) Since $P=0$, $y_3=0$ and $V=1{,}000$, *all* of the pitch should go to the *visbreaker unit* to yield 800 gallons per hour of tar (remember, $T=0.8V$).

(2) Since $F=500$, the 800 gallons of tar should then be blended with 500 gallons of flux to obtain commercial fuel oil.

(3) Since $y_1=0$, the minimum viscosity specification is in fact met.

(4) Since $y_2=500$, the minimum gravity specification is exceeded. (In fact, it is exceeded by a shade more than 1.5 points, since the flux part of the blend is 5/13 and has a gravity of 24, while the tar part of the blend is 8/13 and has a gravity of 7, which results in a fuel oil gravity of 13.5+).

(5) And from our functional $Z = 5P + 4V - 3F = 0 + 4{,}000 - 1{,}500 = 2{,}500$, profits are maximized at $25 per hour.

There is no doubt about this constituting the optimal course of action, but even in this simple little case it is by no means obvious that this is so, without first resorting to linear programming methods both to discover and to prove it the optimal course. It is the purpose of the next two chapters to consider something of those methods and their rationale, using still simpler examples of a necessarily abstract nature to illustrate the methods and rationale. After this initial sortie into the abstract, we should be able to return to the blending problem (at the end of Chapter II) and see the true nature of it more clearly. Actual *solution* of this particular problem will be found in the first section of Chapter VI, where it is used to illustrate duality.

CHAPTER II

Elementary Concepts Visualized

In this chapter we shall be largely concerned with laying a geometric foundation in two- and three-dimensional pictures, in order that we may ultimately visualize something of the nature of, and the solution to, linear programming problems. But it is desirable to begin by stating in a completely general form what we now know to be the linear programming problem. A mathematical generalization of our specific blending problem would be:

(1) To solve the following system of m inequalities in n unknowns:[1]

$$\begin{array}{l} a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \leq b_1 \\ a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \leq b_2 \\ \cdots\cdots\cdots\cdots\cdots \\ a_{m1}y_1 + a_{m2}y_2 + \cdots + a_{mn}y_n \leq b_m \end{array}$$

which system we can turn into the following system of m equations in $(n+m)$ unknowns:

$$\begin{array}{llll} a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n + y_{n+1} & & & = b_1 \\ a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n & + y_{n+2} & & = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & & \\ a_{m1}y_1 + a_{m2}y_2 + \cdots + a_{mn}y_n & & + y_{n+m} & = b_m \end{array}$$

this latter system to be solved in such a way as:

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[1] There is a reason for the seemingly perverse insistence upon using the letter y (rather than the more natural letter x) in the preceding chapter, in this chapter, and in many other places in this book. Matrix algebra convention is to let the first subscript of the coefficients represent row number and to let the second subscript represent column number—as is done here. But there is a relationship between linear programming and game theory; and in game theory, as we shall see in Chapters VII to IX, the letter y is customarily associated with Player B and with the matrix of coefficients in such a way as to give us an implicit system of equations in which

$$\sum_{j=1}^{n} a_{ij}y_j \leq \eta \qquad \text{(where } \eta = \text{“the value of the game”)},$$

for all $i=1, \cdots, m$. This is the same general type of inequality as that with which we are here concerned: namely, a system of inequalities in which the left side is $\leq$ the right side, and the jth y is associated with the jth column of the matrix. The letter x is, on the other hand, associated with the *transpose* (explained in Chapter VI) of this “original” matrix of coefficients and involves a system of inequalities in which

$$\sum_{i=1}^{m} a_{ij}x_i \geq \eta, \qquad \text{for all } j = 1, \cdots n$$

so the ith x is associated with the ith row of the *un*transposed matrix. This latter system of inequations is conventionally associated, in game theory, with Player A.

(2) To have all components of $y = (y_1, y_2, \cdots, y_{n+m})$ non-negative (i.e., either zero or positive) and

(3) To maximize (or minimize) some function of y

$$f(y) = c_1y_1 + \cdots + c_ny_n$$

The first number of the subscripts for the coefficients, a, and the subscript for the coefficients, b, merely stand for the *row* number (or equation number) in which those coefficients appear. And the second subscript number of the coefficients, a, simply stands for the *column* number (or particular y) with which those coefficients are identified. Thus, a_{mn} represents the number in the mth row (or equation) associated with the nth y, which naturally appears in the nth column. And b_m simply stands for the number on the right side of the mth equation.

Now the purpose of first steeping ourselves in this symbolic statement of the linear programming problem is not to state an understandable problem in language difficult to understand—for a symbolic statement *is* difficult to grasp, until one has digested the meaning of the symbols. Rather, the purpose is to impress upon our minds that the examples considered here in two equations and three unknowns, and in three equations and four unknowns, could just as readily be examples in a hundred (or in m) equations in two hundred (or in n) unknowns. To confine ourselves to two or three dimensions simply avoids unnecessary and tedious labor, while also keeping us within dimensions we can readily visualize.[2]

---

[2] While we cannot visualize beyond three dimensions, n-dimensional space is merely an abstraction involving a logical extension of concepts and definitions applicable to dimensions within our capacity to visualize. We know from elementary analytical geometry and the Pythagorean theorem, for example, that, if we have two points in two-dimensional space

$$P_1 = (a_1, a_2) = (5, 1)$$
$$P_2 = (b_1, b_2) = (1, 4)$$

the linear distance between P_1 and P_2 can be thought of as the hypotenuse of a right triangle so that

$$d(P_1, P_2) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2} = \sqrt{(5 - 1)^2 + (1 - 4)^2} = 5$$

It is also easy to see that, if our two points happen to be in a three-dimensional figure so that

$$P_1 = (a_1, a_2, a_3)$$
$$P_2 = (b_1, b_2, b_3),$$

the linear distance from P_1 to P_2 will be

$$d(P_1, P_2) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}$$

Beyond three dimensions, therefore, it is logical to extend this definition of distance be-

The easiest way to visualize what is entailed in finding the optimum solution to a set of equations having more unknowns than equations is to begin with a visualization of the solution to a set of equations having the same number of unknowns as equations. We shall get a unique solution to this.[3] Then we can add another unknown without increasing the number of equations. We shall discover that we shall have an infinite number of solutions to this latter problem, and that we can actually visualize these, as long as we do not let the number of requirements equations exceed three. We shall also discover that there is a point where

$$\frac{1}{y_1 + y_2 + y_3} = \text{a maximum,}$$

as well as a point where

$$\frac{1}{y_1 + y_2 + y_3} = \text{a minimum;}$$

and these, too, we shall be able to visualize. By that time, we shall be getting close to the fundamentals of linear programming.

N EQUATIONS IN N UNKNOWNS

Anyone who has had elementary algebra has solved a pair of equations in two unknowns such as

$$\begin{aligned} 4y_1 + y_2 &= 5 \\ 2y_1 + 3y_2 &= 6 \end{aligned}$$

This system has a unique solution, and this solution happens to involve only positive components for $y = (y_1, y_2)$. This unique solution is $y_1 = 9/10$, $y_2 = 7/5$.

We can draw a two-dimensional (plane) picture of this problem and its solution. The three pairs of coefficients associated with y_1 (4, 2) and y_2 (1, 3) and constituting the right sides of the two equations (5, 6) can be thought of as three points in two-dimensional space. So we plot these three points and label them as P_1 and P_2 (to indicate

---

tween two points to

$$d(P_1, P_2) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \cdots + (a_n - b_n)^2},$$

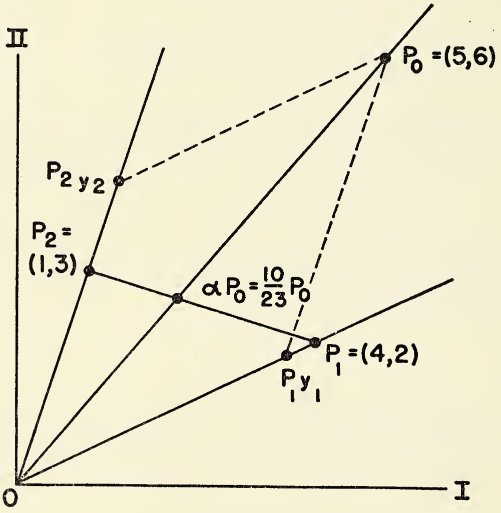
even though we cannot visualize this distance in any natural manner.

[3] Having the same number of equations as variables does not necessarily guarantee this result, but this will be true for the examples we use.

that these two points are associated with y_1 and y_2, respectively) and as P_0, as shown in Diagram 1.

When one solves these equations for y_1 and y_2, what one is really doing is finding the (unique) values for y_1 and y_2 which can be multiplied by P_1 and P_2, respectively, to yield P_0. If we wish, therefore, we can also plot P_1y_1 [which is 9/10 of (4, 2), or (18/5, 9/5)] and P_2y_2 [which is 7/5 of (1, 3), or (7/5, 21/5)]. The sum of these is (25/5, 30/5) = (5, 6) = P_0, as it should; and one can intuitively sense that the parallelogram completed by the broken lines shows our solution is unique: i.e., that the distances from the origin to P_1y_1 and to P_2y_2 must equal, respectively, the distances from P_2y_2 and P_1y_1 to P_0, and only one pair of values for y_1 and y_2 will do this.

DIAGRAM I



Next, suppose we draw a straight line from the origin of our diagram through P_0 and *define*

$$\alpha = \frac{1}{y_1 + y_2}$$

so that, in this case,

$$\alpha = \frac{1}{9/10 + 7/5} = 10/23.$$

This line through P_0 will intersect the straight line joining P_1 and P_2 at αP_0

$$\left(\text{i.e., at the point, } \frac{1}{y_1 + y_2} (P_0), \text{ or at } 10/23\ P_0\right).$$

In words, the point at which the ray through P_0 intersects the line joining P_1 and P_2 is P_0 multiplied by the reciprocal of $y_1 + y_2$ and the more general definition of

$$\alpha = \frac{1}{y_1 + \cdots + y_{n+m}}$$

constitutes a concept of real value in understanding the general nature of solutions to linear programming problems.

We might note one other thing of importance about this diagram. If the ray through P_0 does not intersect the line joining P_1 and P_2, there is no all-non-negative-components-of-y solution to the problem. For example, if P_0 were changed to (5, 2), the ray running from the origin through the new P_0 would lie below the line drawn from the origin through P_1 and could not intersect the line joining P_1 and P_2. This is because a negative component of y enters into the solution of this pair of equations. In fact, with $P_0 = (5, 2)$, the solution is $y = (y_1, y_2) = (13/10, -1/5)$. This is a type of solution *not* permissible within the linear programming framework.

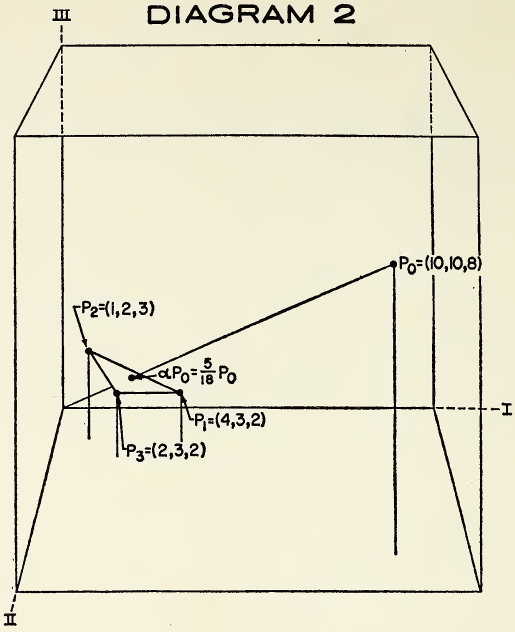
Moving to three equations in three unknowns gives us the same general type of picture, except that it is in one higher dimension. If we solved the system

$$\begin{aligned} 4y_1 + y_2 + 2y_3 &= 10 \\ 3y_1 + 2y_2 + 3y_3 &= 10 \\ 2y_1 + 3y_2 + 2y_3 &= 8, \end{aligned}$$

we should get the unique solution of $y_1 = 9/5$, $y_2 = 4/5$, $y_3 = 1$.

We can depict the four points in three dimensions of $P_1 = (4, 3, 2)$, $P_2 = (1, 2, 3)$, $P_3 = (2, 3, 2)$, and $P_0 = (10, 10, 8)$ in a three dimensional diagram such as Diagram 2. We can think of these points as the heads of four pins standing in an open-front box. The lower left corner at the back of the box represents the origin, the lower back edge is axis 1, the lower edge of the left side is axis 2, and the vertical edge rising from the origin is axis 3. Then the pin the head of which represents P_0, for example, is placed so its point on the bottom of the box is 10 units to the right of the origin and 10 units forward from the lower back edge, while the pin rises 8 units directly above this point,

DIAGRAM 2



so that the pinhead constitutes a point in three-dimensional space representing the coordinate values of (10, 10, 8).

Next we join P_1, P_2, and P_3 with straight lines to give us a two-dimensional triangular plane supported in three-dimensional space by its three pins. Finally, we draw a ray from the origin through P_0, and note that this ray passes through the triangular plane at one point. As we did in our two-dimensional Diagram 1, we could also plot the points (P_1y_1, P_2y_2, and P_3y_3) whose three-dimensional values would naturally sum to $P_0 = (10, 10, 8)$; but this would unnecessarily clutter up an already busy diagram.

The important similarities between this diagram and the preceding two-dimensional one are fairly obvious. For an all-non-negative-components solution for y, the ray through P_0 must pass at a single point through the linear or plane figure delineated by the points representing the coefficients of the components of y—a straight line in Diagram 1 and a triangle in Diagram 2. And in both cases, this point is αP_0, with α being the reciprocal of the sum of the components of y representing the unique solution to the set of simultaneous equations.

While we can push visualization of the entire problem no further than three dimensions, with a mixture of a little algebra and intuition we can see that the conclusion most vital to us holds for all dimensions. We know that in Diagram 1,

$$P_0 = P_1y_1 + P_2y_2 = \sum_{j=1}^{2} P_jy_j$$

and in Diagram 2,

$$P_0 = P_1y_1 + P_2y_2 + P_3y_3 = \sum_{j=1}^{3} P_jy_j.$$

So we can always say that

$$P_0 = \sum_{j=1}^{n=m} P_jy_j,$$

no matter what the dimension of each P_j, i.e., no matter how many rows of coefficients each P_j includes. Consequently, we will always have a single "point" in $n=m$ dimensions representing

$$\alpha P_0 = \alpha \sum_{j=1}^{n=m} P_jy_j,$$

and α will represent the proportion of the "distance" (in the sense indicated in footnote 2) that αP_0 lies from the origin towards P_0. Thus, in Diagrams 1 and 2, αP_0 is 10/23 and 5/18, respectively, of the way towards P_0. And in both cases, as would also be true for any dimension involving an equal number of equations and unknowns, we get but a single point, αP_0. But, as we shall shortly see, when we have more unknowns than equations, we may get an infinite number of αP_0's located on a straight "line," and one of these will be a minimum αP_0 and one will be a maximum αP_0.

M EQUATIONS IN N UNKNOWNS[4]

When we admit more unknowns than we have equations into our system, we take an important step towards understanding the problem of linear programming. We can easily illustrate this by adding a third unknown (and coefficients for it) to the simple example depicted in Diagram 1, to get:

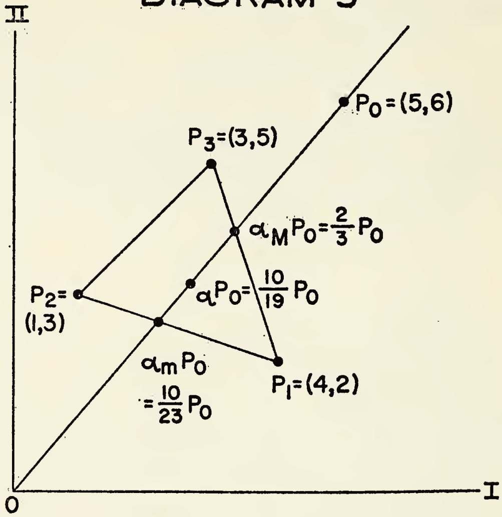
$$4y_1 + y_2 + 3y_3 = 5$$
$$2y_1 + 3y_2 + 5y_3 = 6$$

---

[4] While the typical linear programming problem is initially in terms of requirements inequalities, in the interests of greater simplicity, most of our problems will be in terms of equations. As we have already seen, it is easy to turn the typical inequalities into equations; but this unnecessarily complicates the picture for us by resulting in additional variables.

This is a problem still having a two-dimensional requirements space, because the requirements to be met are still stated in only two equations. But the *solutions* space has been increased to three dimensions, because we now have three unknowns, and if we wanted to depict the solution, $y=(y_1, y_2, y_3)$ as a point in space, we could do so only in a diagram having three coordinates.

DIAGRAM 3


II
$P_0=(5,6)$
$P_3=(3,5)$
$\alpha_M P_0=\frac{2}{3}P_0$
$P_2=(1,3)$
$\alpha P_0=\frac{10}{19}P_0$
$\alpha_m P_0 = \frac{10}{23}P_0$
$P_1=(4,2)$
0
I

One need not be a polished mathematician to see that this pair of requirements can be met in more than one way. We can, for example, set any one of the three unknowns equal to zero, and solve for the remaining two; so there are three solutions involving one zero component for $y=(y_1, y_2, y_3)$. These solutions are:

$$y = (y_1, y_2, y_3) = (0, -7/4, 9/4)$$
$$y = (y_1, y_2, y_3) = (1/2, 0, 1)$$
$$y = (y_1, y_2, y_3) = (9/10, 7/5, 0)$$

The first of these is not acceptable for our purposes, since it involves the negative component, $y_2=-7/4$, but the other two are acceptable. Furthermore, there is an infinite number of solutions involving positive values for all three components. For example, $y=(y_1, y_2, y_3) =(7/10, 7/10, 1/2)$ will satisfy both requirements. How, then, do we choose the "best" solution from the infinite number of solutions?

The best solution will depend on more than the requirements equations; it will also depend on the functional to be maximized (minimized). With the simplifying assumption that we wish to maximize $y_1+y_2+y_3$, the way is clear, if we draw a two-dimensional picture of the problem as is done in Diagram 3. This diagram differs materially from Diagram 1 only in that it has the additional point, $P_3=(3, 5)$, but this additional point has drastically altered the nature of the solution. The unique solution of Diagram 1 is still *a* solution; in fact, it is *the* solution which *minimizes* α—and, therefore, which maximizes $(y_1+y_2+y_3)=9/10+7/5+0)=23/10$, since α is the reciprocal of $(y_1+y_2+y_3)$.

But the solution $y=(y_1, y_2, y_3)=(1/2, 0, 1)$ is also a solution; and this is *the* solution which *maximizes* α—and, therefore, *minimizes* $(y_1+y_2+y_3)=(1/2+0+1)=3/2$. And the entire line segment lying between $\alpha_m P_0$ and $\alpha_M P_0$ represents an infinite number of solutions with all three components of y positive; but each of these will have a sum for the components of y which is smaller than for those of α_m and larger than for those of α_M. Thus, $y=(y_1, y_2, y_3)=(7/10, 7/10, 1/2)$ means an αP_0 of $10/19\ P_0$, and $23/10>19/10>3/2$. If the functional in our linear programming problem told us to maximize (or minimize)

$$f(y) = y_1 + y_2 + y_3,$$

we could clearly put our hands on the answer to it, in this problem, in a hurry: it would be the y components associated with α_m (or α_M).

Nor would the introduction of additional unknowns into our two equations result in any new complications. If, for example, we had five unknowns, Diagram 3 would simply show the αP_0 ray passing through a pentagon (and cutting it at just two points), instead of through a triangle. But, of course, it is always true that, for an all-non-negative-components solution of y to exist, P_0 must be so situated that the αP_0 ray does not fail to cut through the convex set of points delineated by the triangle, pentagon, or other figure described by the points (i.e., columns of coefficients) on the left side of the equations.

Moving to a three-dimensional requirements space and a four-dimensional solutions space results in similar conclusions. This we can easily visualize by merely adding another unknown to the problem illustrated in Diagram 2, to get the system:

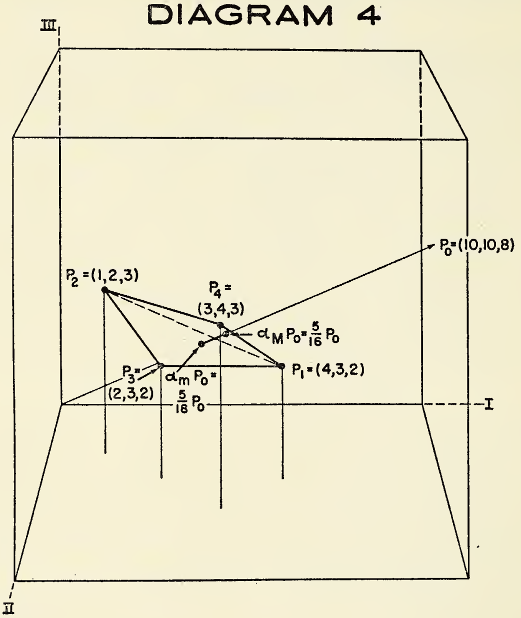
$$\begin{aligned} 4y_1 + y_2 + 2y_3 + 3y_4 &= 10 \\ 3y_1 + 2y_2 + 3y_3 + 4y_4 &= 10 \\ 2y_1 + 3y_2 + 2y_3 + 3y_4 &= 8 \end{aligned}$$

This system has two solutions involving a zero value for one of the components of y and positive values for the three remaining components. These solutions are:

$$y = (y_1, y_2, y_3, y_4) = (8/5, 3/5, 0, 1)$$
$$y = (y_1, y_2, y_3, y_4) = (9/5, 4/5, 1, 0)$$

Diagram 4 shows the points P_1, P_2, P_3, and P_4 in the same three-dimensional setting as was used in Diagram 2. If all four of these points were joined by straight lines (P_3 and P_4 are purposely not joined in order to avoid unnecessarily complicating the picture), they would form a three-dimensional figure in the form of a tetrahedron suspended in the box.

DIAGRAM 4



With P_3 and P_4 not joined, we can imagine ourselves looking into the jaws of a sort of triangular-shaped trap. The hinge of the trap is the broken line joining P_1 and P_2, the upper jaw of the trap (the bottom of which we are seeing) is formed by the triangle $P_1P_2P_4$, and the lower jaw of the trap (the top of which we are seeing) is formed by the triangle $P_1P_2P_3$. As the ray through P_0 travels up from the origin, it disappears from our view behind the lower jaw, and then it reappears as it first pierces the tetrahedron on the plane triangular surface of $P_1P_2P_3$. This point is $\alpha_m P_0 = 5/18\ P_0$. As we know, α_m implicitly carries the unique solution for $y = (y_1, y_2, y_3) = (9/5, 4/5, 1)$ of our problem in Diagram 2; although, in the expanded problem of Diagram 4, we really have that $y = (y_1, y_2, y_3, y_4) = (9/5, 4/5, 1, 0)$. This is the point at which $(y_1 + y_2 + y_3 + y_4) = (9/5 + 4/5 + 1 + 0) = 18/5$ is *maximized.*

As the ray continues up towards P_0, we see it disappear (briefly) at the point where it cuts through the bottom side of the plane triangular surface described by $P_1P_2P_4$. This is the point of $\alpha_M P_0 = 5/16\ P_0$; and this is the point where

$$\alpha = \frac{1}{y_1 + y_2 + y_3 + y_4} = \frac{1}{8/5 + 3/5 + 0 + 1} = 5/16,$$

or the point at which the sum of $(y_1 + y_2 + y_3 + y_4)$ is *minimized* at 16/5. As we might now surmise, the line segment of the ray through P_0 which lies between $\alpha_m P_0$ and $\alpha_M P_0$ represents the infinite number of solutions involving positive values for all four components of y; but, in each of these cases, the sum of these four components will be smaller than those involved in α_m and larger than those involved in α_M.

We might note one further rather obvious similarity between Diagrams 2 and 4. In Diagram 2, the ray through P_0 does not cut the line joining P_2 and P_3; and, in Diagram 4, the ray does not cut either of the two triangular planes formed by P_2, P_3, and P_4 and by P_1, P_3, and P_4. In all cases, this is because a solution involving only the components of y associated with these particular points would result in at least one negative component—an unacceptable solution for linear programming purposes.

THE BLENDING PROBLEM AGAIN

Let us now reconsider the blending problem of Chapter I. To do so will serve three useful purposes: (1) it will provide us with an illustration of what we have just considered somewhat in the abstract; (2) it will emphasize how cumbersome even a relatively simple problem in linear programming can be; (3) it will serve to illustrate the nature and importance of the next step towards an understanding of the rationale underlying the solution of linear programming problems.

In originally stating the blending problem, we used certain mnemonic symbols in the equations: P for pitch, V for visbreaker feed, and F for flux. To these we added three giveaway factors (i.e., factors involving actual or potential relaxation in the requirements equations' stringency), and we labeled these as y_1, y_2, y_3. Visualizing this problem will be less confusing, if we now use y components throughout, so that:

y_1 = pitch (formerly labeled P)
y_2 = visbreaker feed (formerly labeled V) 1000
y_3 = flux (formerly labeled F) 500
y_4 = viscosity giveaway factor (formerly labeled y_1)
y_5 = gravity giveaway factor (formerly labeled y_2) 500
y_6 = materials giveaway factor (formerly labeled y_3)

Stating the problem in these symbols, we have:

(1) Maximize:

$$z = 5y_1 + 4y_2 - 3y_3$$

(2) Subject to the requirements that:

$$\begin{aligned} 2y_1 + y_2 - 2y_3 + y_4 \qquad\qquad &= 0 \\ y_1 + y_2 - 3y_3 \qquad + y_5 \qquad &= 0 \\ y_1 + y_2 \qquad\qquad\qquad + y_6 &= 1{,}000 \end{aligned}$$

This restatement of the equations makes it possible to depict the problem in three-dimensional points and automatically know which

$$\overset{6}{\underset{=1}{P_j}}$$

represents the column of three coefficients associated with which

$$\overset{6}{\underset{=1}{y_j}}.$$

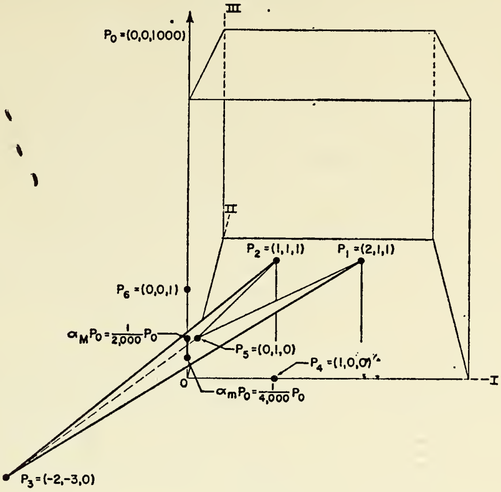
Thus, for example, we shall immediately know that P_1 represents the three coefficients of y_1, and so on.

The six points representing the six sets, or column vectors, of three coefficients associated with

$$\overset{6}{\underset{=1}{y_j}}$$

are plotted in Diagram 5. In this diagram, the origin has been placed at the lower left *front* corner, so that higher values for axis 2 now run

DIAGRAM 5



-2 2
2 -3
0
-5
3

from front to back, rather than the reverse (as was the case in Diagrams 2 and 4). This change in the position of the origin makes it easier to visualize this particular problem. It should be noted that P_0 lies directly above the origin, *so that the vertical axis is also the ray through* P_0. But P_0 itself does not show in the diagram, since the diagram would have to be many times taller to include P_0, unless we reduced the scale to a point where we could not visualize the important things. Note that we also have another new wrinkle in this diagram: a point (P_3) with two negative coordinates, although this introduces no new problems.

Now an optimum solution(s) to this problem, if a solution exists (as we have already been told it does), will exist in m positive components of y (3 components, in this case), with the remaining $(n+m-m)$ components $(3+3-3=3$, in this case) being zero.[5] The

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[5] Even this cannot be accepted as a universal truth. One of several complications, the resolutions of which are considered in Chapters IV and V, is what to do when we bump into a case of so-called degeneracy—a case in which we encounter a solution involving less than m positive components for y, with the remaining components being zero. For example, if we changed the P_0 of Diagram 4 from (10, 10, 8) to (10, 10, 10), the ray through P_0 would *not* go through the tetrahedron, but it *would* just touch the common side of triangles $P_1P_2P_3$ and $P_1P_2P_4$, giving a solution in y_1 and y_2 alone. Unless the solutions space involved more than four unknowns (and, therefore, more than the four points,

$$\overset{4}{\underset{j=1}{P_j}}$$

of Diagram 4) this would be the only solution to our problem, even though that solution does involve but two positive components for y.

cumbersome nature of even this simple problem is apparent when we realize that there are 20 different ways in which we can combine the six components of y into groups of three: in other words, there are 20 different sets of three equations in three unknowns to be solved.

All but two of these sets result in inconsistency, negative coordinates, or more than three zero (and, therefore, less than three positive) components of y. The two exceptions are those which involve: (1) setting y_2, y_4, and y_6 equal to zero, thus solving for y_1, y_3, and y_5; (2) setting y_1, y_4, and y_6 equal to zero, thus solving for y_2, y_3, and y_5. In other words, four of our six points are involved, and we join these four points (except for P_1 and P_2) in Diagram 5.

This picture is strikingly similar to that of Diagram 4. Again we are looking into the jaws of a (somewhat less aesthetically appealing) trap, with the hinge in back represented by the broken line joining P_3 and P_5. The ray through P_0 (the vertical axis, in this case) appears through the top of the bottom jaw at

$$\alpha_m P_0 = \frac{1}{4{,}000} P_0 = (0, 0, 1/4)$$

and disappears (briefly) through the bottom of the top jaw at

$$\alpha_M P_0 = \frac{1}{2{,}000} P_0 = (0, 0, 1/2).$$

A little reflection reveals clearly that this visualization process is good only for what we have intended it: namely, to grasp more easily the fundamental nature of linear programming problems and their solutions. The visualization is certainly not a substitute for solving, or even for finding the best point of initial attack on, a linear programming problem. We can depict most of the more salient points in the blending problem in a perspective fairly easy to grasp. But we can do this only because we started with a knowledge of all the answers. One can readily imagine the intricate picture we should have on our hands, if we made all possible straight-line connections between the six points in order to show all of the 20 possible triangles!

On the other hand, armed with all 20 answers in advance, one can visualize that:

(1) the P_0 ray does not touch *any part* of 7 of the 20 possible triangles—$P_1P_2P_3$, $P_1P_2P_4$, $P_1P_2P_5$, $P_1P_3P_4$, $P_1P_4P_5$, $P_2P_3P_4$, and $P_2P_4P_5$;

(2) the P_0 ray touches 10 of the 20 possible triangles only at the P_6 vertex—$P_1P_2P_6$, $P_1P_3P_6$, $P_1P_4P_6$, $P_1P_5P_6$; $P_2P_3P_6$, $P_2P_4P_6$, $P_2P_5P_6$; $P_3P_4P_6$, $P_3P_5P_6$; and $P_4P_5P_6$;

(3) the P_0 ray goes "through" one of the triangles only at the coordinates point of $(0, 0, 0) - P_3P_4P_5$.

This leaves only two other triangles, which are the two depicted in Diagram 5.

The triangle formed by P_1, P_3, P_5, in which

$$\alpha_m P_0 = \frac{1}{4{,}000} P_0$$

lies, represents positive y components of $(y_1, y_3, y_5) = (1{,}000, 1{,}000, 2{,}000)$.[6] The triangle formed by P_2, P_3, P_5, in which

$$\alpha_M P_0 = \frac{1}{2{,}000} P_0$$

lies, represents positive y components of $(y_2, y_3, y_5) = (1{,}000, 500, 500)$, and we read in Chapter I that this latter solution is the maximizing one. Since α_M represents the solution which *minimizes* $(y_1 + \cdots + y_{n+m})$, if it is true that the (y_2, y_3, y_5) solution is the maximizing one, it must follow that the solution which maximizes $(y_1 + \cdots + y_{n+m})$—namely, the solution represented by α_m—is not necessarily the solution which maximizes the functional.[7]

This means that what we get from α_m and α_M is primarily an insight into the whole range of *feasible* solutions to the linear programming problem, rather than a certain indication of the solution which will maximize the functional. We can rightly infer from Diagram 5 that there is a solution involving positive values for (y_2, y_3, y_5), a solution involving positive values for (y_1, y_3, y_5), and an infinite number of solutions involving positive values for (y_1, y_2, y_3, y_5) represented by the line segment between $\alpha_M P_0$ and $\alpha_m P_0$. We cannot, however, know the solution which maximizes the functional, without considering the functional itself.

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[6] This means blending all 1,000 of the pitch with 1,000 flux; and $y_5 = 2{,}000$ would mean giving away 4 points of gravity (i.e., making the fuel oil gravity 16, instead of the minimum 12).

[7] Perhaps it should be noted that we could also form a tetrahedron by joining P_3, P_4, P_5, and P_6. Here α_M is at P_6 and equals 1/1,000 ($y_6 = 1{,}000$, and all other components equal zero). Since y_6 = the pitch we do *not* run, this says we could minimize "profits" (this is, of course, not the economist's concept of profits, since fixed costs continue even at a zero rate of operation) at zero by doing nothing. And α_m is at the coordinates point (0, 0, 0), with a value of 1/0, which is meaningless, as is also the implicit "solution" involving flux, viscosity giveaway, and gravity giveaway.

Determination of the solution optimizing the functional is the next step towards an understanding of the rationale underlying the solution of linear programming problems. The answer to this question could, of course, be illustrated with reference to this same blending problem. But a properly chosen hypothetical case is better calculated to offer the maximum understanding with a minimum of unnecessary complications.

CHAPTER III

The Simplex Method

Perhaps the best-known method of solving a linear programming problem is the simplex method. This method, due to G. B. Dantzig,[1] at each stage indicates (among other things) whether the solution is optimal, and where the solution can be bettered if it is not optimal. That is, it does these things unless we first run into a solution involving fewer positive components of y than we have equations—and A. Charnes has devised a method for coping with this contingency.[2]

While the simplex method can be reduced to a set of routine steps requiring nothing more than arithmetic, our principal interest lies in the rationale behind this routine. This rationale is difficult to follow, unless we first learn a little about three matrix algebra concepts. Armed with these, we can then work through the algebraic rationale of a simple example, generalize this algebraic rationale, and end by displaying all this in tabular form in the so-called simplex tableau.

MATRIX ALGEBRA

Even at an elementary level, the general rules of matrix algebra are numerous and somewhat complicated. To acquire the insight we need, however, it is enough to understand but three of these and to be able to apply them to a small matrix. These three concepts are determinants, matrix multiplication, and the inverse matrix.

We can acquire some understanding of these concepts through the following simple problem, which we have previously met:

$$\begin{aligned} 4y_1 + y_2 &= 5 \\ 2y_1 + 3y_2 &= 6 \end{aligned} \qquad \text{or, more generally,} \qquad \begin{aligned} a_{11}y_1 + a_{12}y_2 &= b_1 \\ a_{21}y_1 + a_{22}y_2 &= b_2 \end{aligned}$$

If we multiplied the first equation by 3 (the a_{22} coefficient) and the second equation by 1 (the a_{12} coefficient), y_2 would cancel out, if we then subtracted the second equation from the first equation, after which we could solve for y_1. In other words:

$$y_1 = \frac{3 \cdot 5 - 1 \cdot 6}{3 \cdot 4 - 1 \cdot 2} = 9/10; \quad \text{or,} \quad y_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{22}a_{11} - a_{12}a_{21}}$$

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[1] T. C. Koopmans (ed.), *Activity Analysis of Production and Allocation* (New York: John Wiley and Sons, Inc., 1951), Chapter XXI.

[2] A. Charnes, "Optimality and Degeneracy in Linear Programming," *Econometrica,* 1952, pp. 160–172.

Similarly, it can be seen that

$$y_2 = \frac{4\cdot 6 - 2\cdot 5}{4\cdot 3 - 2\cdot 1} = 7/5; \qquad \text{or,} \quad y_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{21}a_{12}}$$

We can express both the numerator and the denominator for both y_1 and y_2 in terms of 2×2 determinants, a determinant being a function of a square matrix (or array of numbers) which can be "evaluated" into a single number in solving simultaneous equations:

$$y_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}; \qquad y_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \qquad (a_{11}a_{22} - a_{12}a_{21}) \neq 0$$

We can learn two important things from these last expressions for y_1 and y_2. In the first place, we can immediately infer the general rule for *evaluating* (i.e., finding the value of) a 2×2 determinant. We have just seen above that the denominator, for example, of y_1 (or of y_2) is $(a_{11}a_{22} - a_{12}a_{21})$. If this is so, then by definition, the value of a 2×2 determinant must be the cross product of its principal diagonal $(a_{11}a_{22})$ minus the cross product of its other diagonal $(a_{12}a_{21})$. A more complicated and more general rule also exists for evaluating determinants of any order, but this is unnecessary for us. The important thing is that, from this example of evaluating a determinant of order 2, we can readily grasp the logic underlying determinant-evaluation rules.

Secondly, we see that y_1 and y_2 have the same denominator, *and this denominator is the determinant of the matrix of the coefficients of* y_1 *and* y_2. This means, in effect, that we are *multiplying* the *determinant* of the matrix *numerators* of both y_1 and y_2 by the *reciprocal* of the *determinant* of the *matrix of coefficients for* y_1 *and* y_2. This is the heart of a concept for which we shall have real use: the inverse matrix.[3] But, before we can consider this, we must learn one other rule: the rule for *matrix multiplication.*

The rationale for the rule of matrix multiplication is rooted in the same general setting of elementary algebra as is the rule for determinant evaluation. This being so, perhaps we can accept the rule for matrix multiplication on faith, since a demonstration of its rationale would be more extensive than seems justifiable for our purposes.

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[3] In matrix algebra, there is no such operation as *dividing* one matrix by another matrix; but *multiplication* by the *inverse* of the denominator matrix accomplishes, as we shall see, the equivalent of this familiar operation.

To state the rule in its most terse (and most frightening) form, if we pre-multiply a matrix B by a matrix A, the product of AB will also be a matrix C each of whose elements c_{ij} is obtained by the rule:

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

This compact notation has much to recommend it, but it is certain to be meaningless to a non-mathematician, until he has digested what it means in words. Let us consider, then, what this symbolic rule says in words.

Suppose both matrix A and matrix B are 2×2 matrices, and we want to find c_{11} (the element in the first row and the first column of matrix C), so $i=1$ and $j=1$. Now the subscript k stands for the kth *column* in A, but for the kth *row* in B. Since both A and B are 2×2 matrices, k stands for both 1 and 2 (but not, of course, simultaneously). Therefore, $c_{11}=a_{11}b_{11}+a_{12}b_{21}$; and, if the c_{ij} we want is c_{12}, then $c_{12}=a_{11}b_{12}+a_{12}b_{22}$. In yet different words, to get the element in the *first* row and the *first* column (c_{11}) of C, we multiply each element in the *first row* of A into its corresponding (first or second) element in the *first column* of B, and add the two. And to get the element in the *first* row and the *second* column (c_{12}) of C, we multiply each element in the *first* row of A into its corresponding element in the *second* column of B, and add the two. Thus, we can say that the matrix multiplication rule is one of multiplying row into column.[4]

To illustrate this, consider a concrete example of multiplying two matrices for which we shall have a future use:

$$\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \times \begin{bmatrix} 3/10 & -1/10 \\ -1/5 & 2/5 \end{bmatrix}$$

$$= \begin{bmatrix} (12/10 - 1/5) & (-4/10 + 2/5) \\ (6/10 - 3/5) & (-2/10 + 6/5) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The product of our two matrices thus results in the matrix on the far right—a matrix with ones in the principal diagonal and zeros every-where else. This matrix I is the unit matrix (sometimes called the identity matrix). Furthermore, the second matrix from the left is the *inverse matrix* of the far left matrix; and, in the above illustration of matrix multiplication, we also have an illustration of the nature of an inverse matrix.

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[4] A little reflection makes it clear that this is a law which has no exact counterpart in elementary algebra. The familiar commutative law of multiplication postulates that $ab=ba$. But, in matrix algebra, $AB\neq BA$, i.e., which matrix is the multiplier and which the multiplicand *does* affect the product matrix, C. Nor can we multiply two matrices, unless the multiplier has the same number of *columns* as the multiplicand has *rows*.

The inverse matrix obviously has a counterpart in elementary algebra, which latter says that the inverse of a number, a, is $a^{-1}=1/a$, so that $a \cdot a^{-1}=1$. Similarly, the inverse of matrix A is $A^{-1}=I/A$, so that $A \cdot A^{-1}=I$, I being the identity (or unit) matrix.[5]

Now, from our earlier expressions of y_1 and y_2 in the form of a 2×2 determinant divided by a 2×2 determinant, along with the definition of A^{-1}, it follows that

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{I}{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}} \times \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = A^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

and we can check this for the initial system of two equations in this chapter, since the above inverse matrix of specific numbers is, in fact, the inverse matrix of the matrix of coefficients for y_1 and y_2 in that system:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3/10 & -1/10 \\ -1/5 & 2/5 \end{bmatrix} \times \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} (15/10 & -6/10) \\ (-1 & +12/5) \end{bmatrix} = \begin{bmatrix} 9/10 \\ 7/5 \end{bmatrix}$$

We now have a little knowledge of determinant evaluation, and, more importantly, of matrix multiplication and the inverse matrix. We may be a far cry from polished mathematicians, but we should be well enough equipped to understand the rationale for the simplex method of getting to the solution of a linear programming problem in an efficient manner.

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[5] In case it is not yet obvious how one gets each of the unknown elements (call them g_{ij}) of an inverse matrix, in our illustration above we have:

$$\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \times \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Using the matrix multiplication rule, we find, for example, that:

$$4g_{11} + g_{21} = 1$$
$$2g_{11} + 3g_{21} = 0$$

Solving these, we get $g_{11}=3/10$; $g_{21}=-1/5$, and g_{12} and g_{22} can be determined in a similar manner.

SIMPLEX RATIONALE

If we add two more unknowns to the two equations set forth in the first section of this chapter, we shall have one of the simplest possible cases for illustrating the rationale of the simplex method. So let our problem be:

(1) Given the requirements

$$4y_1 + y_2 + 3y_3 + 0y_4 = 5$$
$$2y_1 + 3y_2 + 0y_3 + 4y_4 = 6$$

(2) Maximize the functional

$$f(y) = 2y_1 + y_2 + 2y_3 + y_4$$

This problem is so simple that it can be quickly solved by the most naive of methods, but we want to solve it the hard way to acquire an understanding of the simplex method. For this latter method would, of course, be the easy way, if we were dealing with a problem of large dimensions.

Suppose we decide to make our first solution attempt that of solving for y_1 and y_2, thus setting y_3 and y_4 equal to zero. In linear programming terminology, y_1 and y_2 would be the *basic variables* in this first stage, while y_3 and y_4 would be the *non-basic variables*. In general, however, we always have in our attempted solutions m (i.e., the same number as the number of equations) basic variables and $(n-m)$ non-basic variables.[6] Having reached this decision, we can express our equations as

$$4y_1 + y_2 = 5 - (3y_3 + 0y_4)$$
$$2y_1 + 3y_2 = 6 - (0y_3 + 4y_4),$$

and the solution for y_1 and y_2 would be given by

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A^{-1}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} - A^{-1}\left\{\begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} y_3 + \begin{bmatrix} a_{14} \\ a_{24} \end{bmatrix} y_4\right\}$$
$$= \begin{bmatrix} 3/10 & -1/10 \\ -1/5 & 2/5 \end{bmatrix} \times \begin{bmatrix} 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 3/10 & -1/10 \\ -1/5 & 2/5 \end{bmatrix}$$

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[6] The symbol n is now being used, and will henceforth generally be used, in a different sense from that in Chapter II. In Chapter II, n represented the total number of variables in the inequalities, while $(n+m)$ represented the total number of variables, after these inequalities were turned into equations by adding another variable to each inequality. Henceforth, unless it is evident that we are doing otherwise, the symbol n should be understood to represent the total number of variables, after the linear programming problem requirements have been stated in equation form.

$$\times \begin{bmatrix} 3 \\ 0 \end{bmatrix} y_3 - \begin{bmatrix} 3/10 & -1/10 \\ -1/5 & 2/5 \end{bmatrix} \times \begin{bmatrix} 0 \\ 4 \end{bmatrix} y_4$$

$$= \begin{bmatrix} 9/10 \\ 7/5 \end{bmatrix} - \begin{bmatrix} 9/10 \\ -3/5 \end{bmatrix} \times 0 - \begin{bmatrix} -4/10 \\ 8/5 \end{bmatrix} \times 0$$

Since $y_3 = y_4 = 0$, the last two terms on the right disappear, and the first term on the right gives us what we have previously seen to be the correct answers for y_1 and y_2.

Nevertheless, we have real interest in the last two terms on the right. To see why, let us begin by generalizing the above solution for y_1 and y_2 into an equation of fundamental importance:

$$(1) \quad \begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ y_m \end{bmatrix} = \begin{bmatrix} g_{11} & \cdots & g_{1m} \\ & & \\ & & \\ g_{m1} & \cdots & g_{mm} \end{bmatrix} \times \left\{ \begin{bmatrix} b_1 \\ \cdot \\ \cdot \\ b_m \end{bmatrix} - \left(\begin{bmatrix} a_{1,m+1} \\ \cdot \\ \cdot \\ a_{m,m+1} \end{bmatrix} y_{m+1} + \cdots + \begin{bmatrix} a_{1n} \\ \cdot \\ \cdot \\ a_{mn} \end{bmatrix} y_n \right) \right\}$$

where

$$\begin{bmatrix} g_{11} & \cdots & g_{1m} \\ \cdot & & \\ \cdot & & \\ g_{m1} & \cdots & g_{mm} \end{bmatrix}$$

is, of course, the inverse matrix (having $m \times m$ elements) of the matrix of coefficients for the m basic variables. In effect, this says that what we know is true for a problem in two requirements equations and four variables is equally true for a problem in m requirements equations and n variables.

In equation (1), as well as in the specific counterpart of it for our little problem, we have particular interest in two types of terms:

(a) *The m (2, in our problem) terms we get from pre-multiplying the column vector $b = (b_1, \cdots, b_m)$ by each of the m rows of the inverse matrix.* These, of course, give us the values of the m components of y. For example,

$$[g_{11} \;\; g_{12}] \times \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = [3/10 \;\; -1/10] \times \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$= (15/10 - 6/10) = 9/10 = y_1.$$

To get such expressions down to manageable size, let us define

$$\overset{m}{\underset{u=1}{z_{u0}}} = [g_{u1} \cdots g_{um}] \times \begin{bmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix}.$$

Thus, z_{10} would mean that $u=1$, and is the number we would get by taking the *first* row of the inverse matrix, pre-multiplying each of the m elements in that row into its corresponding element in the column vector of b, and summing. This also means that $z_{u0}=y_u$; hence, in our example, $z_{10}=y_1=9/10$. And $z_{20}=y_2=7/5$.

(b) *The $m(n-m)$ terms* $[2(4-2)=4$, *in our problem*$]$ *we get from pre-multiplying each of the $(n-m)$ column vectors of coefficients associated with the non-basic variables $(y_{m+1}, \cdots, y_n)$ by each of the m rows of the inverse matrix.* Thus,

$$[g_{11}\ g_{12}] \times \begin{bmatrix} a_{14} \\ \cdot \\ a_{24} \end{bmatrix} = [3/10 - 1/10] \cdot \times \begin{bmatrix} 0 \\ \cdot \\ 4 \end{bmatrix} = (0 - 4/10) = -4/10,$$

in our example. To deal with these $m(n-m)$ terms more compactly, let us define

$$\overset{m}{\underset{u=1}{z_{u,m+k}}} = [g_{u1} \cdots g_{um}] \times \begin{bmatrix} a_{1,m+k} \\ \cdot \\ \cdot \\ \cdot \\ a_{m,m+k} \end{bmatrix},$$

where $k=1$ to $(n-m)$. In our example, this merely means that:

$$z_{1,m+1} = z_{13} = 9/10 \qquad z_{1n} = z_{14} = -4/10$$

$$z_{m,m+1} = z_{23} = -3/5 \qquad z_{mn} = z_{24} = 8/5$$

With these definitions, we have, in effect, labeled each of the numbers gotten on the right side of equation (1) from pre-multiplying a column vector (whether it be the b vector or a vector associated with one of the non-basic variables) by a row vector (whether it be the 1st, or any other, row up to, and including, the mth) of the inverse matrix. And the subscripts tell us readily which row of the inverse matrix and which column vector entered into each of these figures. Thus, z_{m0} is the figure resulting from pre-multiplying the 0th column vector (which is the b column) by the mth row of the inverse matrix; and z_{m0} must be the same thing, then, as y_m. And z_{mn} is the result of pre-multiplying the column vector of coefficients associated with y_n by the mth row of the inverse matrix.

Here, again, complete absorption of the meaning of the notation is important to a ready understanding of the nature of all these z's; but, once the meaning of the notation is firmly embedded, the rest is reasonably smooth sailing. By virtue of the definition of the z's, we can state equation (1) in an alternative (and more compact) form

$$\text{(1a)} \qquad \overset{m}{\underset{u=1}{y_u}} = z_{u0} - \sum_{k=1}^{n-m} z_{u,m+k}y_{m+k}.$$

This means, for example, that in our problem we could say

$$y_1 = z_{10} - (z_{13}y_3 + z_{14}y_4)$$
$$y_2 = z_{20} - (z_{23}y_3 + z_{24}y_4)$$

Since y_3 and y_4 equal zero, this may seem like a nonsensical procedure—and it would be, if we were merely trying to find the values of y_1 and y_2. But the purpose of this is more subtle and more important than that of merely finding the values of y_1 and y_2. *We are now in a position to ascertain whether the* $y = (y_1, y_2)$ *solution is optimal.*

Our functional to be maximized is

$$f(y) = 2y_1 + 1y_2 + 2y_3 + 1y_4$$

If we substitute $z_{10} - (z_{13}y_3 + z_{14}y_4)$ and $z_{20} - (z_{23}y_3 + z_{24}y_4)$ for y_1 and y_2, respectively, into our functional, we get

$$\begin{aligned}(f)y &= (2z_{10} + 1z_{20}) + \{[2 - (2z_{13} + 1z_{23})]y_3 + [1 - (2z_{14} + 1z_{24})]y_4\} \\ &= 16/5 + 4/5y_3 + 1/5y_4\end{aligned}$$

This specific result of substituting into the functional can be generalized into a second equation of fundamental importance. It will pay us to see how.

A general statement of our functional we know to be:

$$\text{Maximize } f(y) = c_1y_1 + \cdot\cdot\cdot + c_ny_n$$

Now consider the specific statement of the last paragraph. The first term in parentheses is actually

$$(c_1z_{10} + c_2z_{20}) = \left(\sum_{u=1}^{m=2} c_uz_{u0}\right).$$

What is the general nature of each of the bracketed terms preceding y_3 and y_4? If we carefully traced each step of the substitution into the functional, we should discover that these bracketed expressions amount to:

$$[c_3 - (c_1z_{13} + c_2z_{23})]y_3 + [c_4 - (c_1z_{14} + c_2z_{24})]y_4$$
$$= \sum_{k=1}^{n-m} \left(c_{m+k} - \sum_{u=1}^{m} c_u z_{u,m+k}\right) y_{m+k}$$

We can generalize the functional, then, as:

$$f(y) = \sum_{u=1}^{m} c_u z_{u0} + \sum_{k=1}^{n-m} \left(c_{m+k} - \sum_{u=1}^{m} c_u z_{u,m+k}\right) y_{m+k} \tag{2}$$

Usually, (2) is expressed even more succinctly by defining

$$z_{00} = \sum_{u=1}^{m} c_u z_{u0} \text{ and } z_{0,m+k} = c_{m+k} - \sum_{u=1}^{m} c_u z_{u,m+k}.$$

Then (2) becomes:

$$f(y) = z_{00} + \sum_{k=1}^{n-m} z_{0,m+k} y_{m+k} \tag{2a}$$

An understanding of both the (2) and the (2a) forms is desirable for an understanding of the symbolic terminology customary in a simplex tableau.

Having extended the substitution of

$$\overset{m=2}{\underset{u=1}{y_u}}$$

into our specific functional into the generalized equations of (2) and (2a), let us return once more to our specific substitution, in order to see its relationship to the problem of determining whether our $y=(y_1, y_2)$ solution is optimal. The first expression in ordinary parentheses tells us what $f(y)$ will be, when $y_3=y_4=0$. This yields a result of $z_{00}=(2\cdot 9/10+1\cdot 7/5)=16/5$. The pertinent question is whether we can increase z_{00} by removing either y_1 or y_2 from our basis (by setting one of them equal to zero) and bringing either y_3 or y_4 into the basis to supplant the variable thus removed.

Both y_3 and y_4 are multiplied by expressions enclosed in brackets [ ], and, if the expression associated with (say) y_3 sums to a negative figure, we will only hurt our cause by increasing y_3 from zero to some positive figure; but, if the sum is positive, we can improve the functional by substituting y_3 for either y_1 or y_2. Why this is so will become clear in a moment, when we consider the rule for removing a variable from the basis.

As it happens, both y_3 and y_4 are preceded by bracketed sums that are positive. There is no infallible rule for telling us which of these should now be introduced as a basic variable. Since y_3 is preceded by the bigger coefficient (4/5) of the two, however, we might choose to introduce y_3.

There *is* a rule for telling us which variable to *remove* from the basis, however, *once we have decided on the variable to be introduced.* We can readily see from equation (1a) that, if we increase y_3 from zero to some positive figure, y_3', the effect upon (say) y_1 will be

$$y_1' = z_{10} - z_{13}y_3'$$

In other words, since we know that $y_1 = z_{10}$, an increase in y_3 will decrease y_1 by $z_{13}y_3'$, *if* z_{13} *is positive* (and, of course, y_1 will be increased by $z_{13}y_3'$, if z_{13} is negative). Since we must not allow y_1 to fall below zero, we cannot safely permit y_3' to be larger than z_{10}/z_{13}. For this reason, we remove from the basis that particular component of

$$\overset{m}{\underset{u=1}{y_u}}$$

for which the positive ratio $z_{u0}/z_{u,m+h}$ is smallest [the $(m+h)$ portion of the subscript being identified with the particular component of y we have decided to bring into the basis (y_3, in our case)].

In our example, the smallest positive z_{u0}/z_{u3} ratio is, in fact, that of z_{10}/z_{13}. We know, therefore, having previously decided to bring y_3 into the basis, that y_1 is the variable to remove. As removing y_1 from the basis means making $y_1' = 0$, it follows that

$$z_{10} - z_{13}y_3' = 0,$$

so $y_3' = z_{10}/z_{13} = 1$. And $y_2' = z_{20} - z_{23}y_3' = 7/5 + 3/5 = 2$.

This clearly represents an improvement in our functional

$$f(y) = 2y_1 + 1y_2 + 2y_3 + 1y_4$$

At the end of the first stage, we had

$$f(y) = 2 \times 9/10 + 1 \times 7/5 + 2 \times 0 + 1 \times 0 = 16/5,$$

and we now have, at the end of the second stage,

$$f(y) = 2 \times 0 + 1 \times 2 + 2 \times 1 + 1 \times 0 = 4.$$

Does this maximize our functional?

If we went through the process we did above of substituting in the functional for the basic variables (now y_2 and y_3) expressions in terms of the z's, y_1, and y_4, our functional would become:

$$f(y) = (1z_{20} + 2z_{30}) + \{[2 - (1z_{21} + 2z_{31})]y_1 + [1 - (1z_{24} + 2z_{34})]y_4\}$$

We know the values of z_{20} and z_{30} (since they are also the y_2' and y_3' we just computed above), but we do not know the values of the other z's.

We can, of course, get the values of these other z's by computing the inverse matrix of the matrix of coefficients of the new basic variables (y_2 and y_3), and pre-multiplying the appropriate row vectors of the inverse matrix into the appropriate column vectors of the new non-basic variables (y_1 and y_4). We needed this process to get under way on the first stage, and this *is* the rationale underlying the z's at any stage. But, just as we found a handy way of calculating y_2' and y_3', once the first stage was completed, there are equally handy algorithms for computing the missing z's, and these algorithms, along with their derivations, are given in the appendix at the end of this chapter. In trying to answer the question of whether we have now maximized our functional, however, it is logical to introduce the notion of the simplex tableau. For it is clear that, in solving a linear programming problem, we are involved in an iterative process: going from solution to better solution to (eventually) an optimal solution.

Even with algorithms to speed the transitions, to shunt us away from blind alleys, to help point the way towards the more promising avenues, and to announce to us when we have arrived, the optimal solution will rarely come to us as a blinding revelation. It is, therefore, important to have an orderly tabulation of successive solutions. This makes it possible to see clearly where we have been, whether we have any place left to go, and, if so, the alternative routes remaining open to us.

THE SIMPLEX TABLEAU

Table I is a so-called *extended simplex tableau*, and it is divided into three parts, each of which includes the pertinent information for one of the three solutions involved in reaching the optimal solution to our simple problem. Thus, the top third is concerned with the first stage of the simple problem we considered in detail.

The first and second columns list the coefficients

$$\left(\text{i.e., the } \overset{m}{\underset{u=1}{c_u}} \text{ from the } \textit{functional} \text{ equation}\right)$$

TABLE I

			Basic Variables		Non-basic Variables	
			$y_u=y_r=y_1$	$y_u=y_m=y_2$	$y_{m+1}=y_{m+h}=y_3$	$y_{m+k}=y_n=y_4$
			$c_u=c_1=\underline{\underline{2}}$	$c_u=c_m=c_2=\underline{\underline{1}}$	$c_{m+1}=c_3=\underline{\underline{2}}$	$c_n=c_4=\underline{\underline{1}}$
$c_u=c_1=\underline{\underline{2}}$	$y_u=y_r=y_1$	$z_{u0}=z_{r0}=z_{10}=\underline{\underline{9/10}}$	$z_{uu}=z_{11}=\underline{\underline{1}}$	$z_{um}=z_{12}=\underline{\underline{0}}$	$z_{u,m+1}=z_{r,m+h}=z_{13}=\underline{\underline{9/10}}^*$	$z_{un}=z_{rn}=z_{14}=-\underline{\underline{4/10}}$
$c_m=c_2=\underline{\underline{1}}$	$y_m=y_2$	$z_{m0}=z_{20}=\underline{\underline{7/5}}$	$z_{mu}=z_{21}=\underline{\underline{0}}$	$z_{mm}=z_{22}=\underline{\underline{1}}$	$z_{m,m+1}=z_{m,m+h}=z_{23}=-\underline{\underline{3/5}}$	$z_{mn}=z_{24}=\underline{\underline{8/5}}$
		$z_{00}=\sum_{u=1}^{m} c_u z_{u0}$ $=2z_{10}+1z_{20}=\underline{\underline{16/5}}$	$z_{01}=\left(c_1-\sum_{u=1}^{m} c_u z_{u1}\right)$ $=[2-(2z_{11}+1z_{21})]=\underline{\underline{0}}$	$z_{0m}=z_{02}=\left(c_m-\sum_{u=1}^{m} c_u z_{um}\right)$ $=[1-(2z_{12}+1z_{22})]=\underline{\underline{0}}$	$z_{0,m+1}=z_{03}$ $\left(c_{m+1}-\sum_{u=1}^{m} c_u z_{u,m+1}\right)$ $=[2-(2z_{13}+1z_{23})]=\underline{\underline{4/5}}$	$z_{0n}=z_{04}$ $=\left(c_n-\sum_{u=1}^{m} c_u z_{un}\right)$ $=[1-(2z_{14}+1z_{24})]=\underline{\underline{1/5}}$
			y_3	y_2	y_1	y_4
			$c_3=\underline{\underline{2}}$	$c_2=\underline{\underline{1}}$	$c_1=\underline{\underline{2}}$	$c_4=\underline{\underline{1}}$
$c_3=\underline{\underline{2}}$	y_3	$z_{30}=\underline{\underline{1}}$	$z_{33}=\underline{\underline{1}}$	$z_{32}=\underline{\underline{0}}$	$z_{31}=\underline{\underline{10/9}}$	$z_{34}=-\underline{\underline{4/9}}$
$c_2=\underline{\underline{1}}$	y_2	$z_{20}=\underline{\underline{2}}$	$z_{23}=\underline{\underline{0}}$	$z_{22}=\underline{\underline{1}}$	$z_{21}=\underline{\underline{2/3}}$	$z_{24}=\underline{\underline{4/3}}^*$
		$z_{00}=2z_{30}+1z_{20}=\underline{\underline{4}}$	$z_{03}=\underline{\underline{0}}$	$z_{02}=\underline{\underline{0}}$	$z_{01}=-\underline{\underline{8/9}}$	$z_{04}=\underline{\underline{5/9}}$
			y_3	y_4	y_1	y_2
			$c_3=\underline{\underline{2}}$	$c_4=\underline{\underline{1}}$	$c_1=\underline{\underline{2}}$	$c_2=\underline{\underline{1}}$
$c_3=\underline{\underline{2}}$	y_3	$z_{30}=\underline{\underline{5/3}}$	$z_{33}=\underline{\underline{1}}$	$z_{34}=\underline{\underline{0}}$	$z_{31}=\underline{\underline{4/3}}$	$z_{32}=\underline{\underline{1/2}}$
$c_4=\underline{\underline{1}}$	y_4	$z_{40}=\underline{\underline{3/2}}$	$z_{43}=\underline{\underline{0}}$	$z_{44}=\underline{\underline{1}}$	$z_{41}=\underline{\underline{1/2}}$	$z_{42}=\underline{\underline{3/4}}$
		$z_{00}=2z_{30}+1z_{40}=\underline{\underline{29/6}}$	$z_{03}=\underline{\underline{0}}$	$z_{04}=\underline{\underline{0}}$	$z_{01}=-\underline{\underline{7/6}}$	$z_{02}=-\underline{\underline{5/12}}$

of the *basic* variables and the "names" of the basic variables in that stage, respectively. The third column contains the

$$\mathop{z_{u0}}\limits_{u=1}^{m},$$

which we know is the same thing as the values of the

$$\mathop{y_u}\limits_{u=1}^{m},$$

i.e., the values, in that stage, of the basic variables. The figure at the bottom of this column shows the value of the functional, z_{00}, and is, of course, the figure obtained by multiplying each z_{u0} by its associated c_u and summing them algebraically, i.e.,

$$z_{00} = \sum_{u=1}^{m} c_u z_{u0}.$$

At the top of the next four columns appear the "names" of *all* variables, the first two of these representing the basic variables, and the last two the non-basic

$$\left(\mathop{y_{m+k}}\limits_{k=1}^{n-m}\right)$$

variables. Underneath each of these four "names" is listed the associated coefficient (c_n) from the functional. The variables bear not only their specific names (e.g., y_1) but their general names[7] as well (e.g.,

$$\mathop{y_u}\limits_{u=1}^{m}$$

---

[7] The general "names" are dropped after the first stage, with only specific subscripts being used. This is done, partially, because it should be enough, for illustrative purposes, to show the general names but once. The general "names" of the first stage apply, however, in succeeding stages, *only if we re-name our variables.* For example, when we introduce y_3 into the basis, we can no longer say that the basic variables are

$$\mathop{y_u}\limits_{u=1}^{m},$$

because the subscript of y_3 is really $(m+1)=(2+1)$. So we would have either to say that the basic variables in the second stage are

$$\mathop{y_u}\limits_{u=2}^{m+1}$$

or to re-name our variables.

for the basic variables, and

$$\overset{n-m}{\underset{k=1}{y_{m+k}}}$$

for the non-basic variables). And, in two cases, they bear two general names $(y_u = y_r)$ for the basic variable to be removed from the basis in the second stage, and $(y_{m+1} = y_{m+h})$ for the non-basic variable to be introduced into the basis in the second stage.

The z's in the non-basic-variables columns are old friends, and we need only make one comment about them. One of them is asterisked in each stage (z_{13}, for example, in the first stage). This is the so-called *pivot*—the point of juncture of the variable to be introduced into, and the variable to be removed from, the basis. Thus, z_{13} has a subscript telling us that y_1 is to be removed and y_3 is to be introduced in the second stage.

The figure at the bottom of each non-basic-variable column is obtained by multiplying each z in that column by the *basic* variable coefficient (from the far left column) in the same row ($c_1 \cdot z_{13}$ and $c_2 \cdot z_{23}$, for example), summing these cz's, and then subtracting that sum from the functional-equation coefficient near the top of each column. Thus for each of these columns we get at the bottom a figure

$$\overset{n-m}{\underset{k=1}{z_{0,m+k}}} = \left(c_{m+k} - \sum_{u=1}^{m} c_u z_{u,m+k}\right).$$

As we already know, if any one of these bottom figures is positive, we can increase z_{00} by next introducing into the basis the variable whose "name" is at the top of that column. Thus, both y_3 and y_4 are possibilities at the end of the first stage.

In the second stage of the tableau, we move y_3 to the position formerly held by y_1, and we move y_1 to the position formerly held by y_3. The various new z's are easily calculated from our algorithms, and we see from z_{03} and z_{04} that y_4 is the only non-basic variable that can be profitably brought into the basis, while y_2 should be removed. Thus, z_{24} is the new pivot.

Finally, replacing y_2 with y_4 as a basic variable and again calculating the z's, we see that both z_{01} and z_{02} are negative. This tells us that we have reached the optimum solution, and we see that this optimum entails $z_{00} = 29/6$.

THE CASE OF $A=A^{-1}=I$

Solution of a linear programming problem can be simplified even further, if the constraints are such that the inverse matrix, A^{-1}, of the initial stage can be made equal to the unit matrix, I. The problem we have treated so laboriously, in the interests of understanding the underlying rationale, can, in fact, easily be turned into such a problem.

If we divide our first constraint through by 3, and our second constraint through by 4, we get:

$$(4/3)y_1 + (1/3)y_2 + y_3 + 0y_4 = 5/3$$
$$(1/2)y_1 + (3/4)y_2 + 0y_3 + y_4 = 3/2$$

The obvious first stage to this problem is to let $y_1=y_2=0$. The matrix of coefficients for our basic variables (y_3, y_4) is now

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which is, itself, a unit matrix. The inverse of a unit matrix is, however, also a unit matrix, so we have:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} g_{13} = 1 & g_{14} = 0 \\ g_{23} = 0 & g_{24} = 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

From equation (1) we can immediately say that

$$\begin{bmatrix} y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left\{ \begin{bmatrix} 5/3 \\ 3/2 \end{bmatrix} - \begin{bmatrix} 4/3 \\ 1/2 \end{bmatrix} y_1 - \begin{bmatrix} 1/3 \\ 3/4 \end{bmatrix} y_2 \right\}$$
$$= \begin{bmatrix} 5/3 \\ 3/2 \end{bmatrix} - \begin{bmatrix} 4/3 \\ 1/2 \end{bmatrix} \times 0 - \begin{bmatrix} 1/3 \\ 3/4 \end{bmatrix} \times 0 = \begin{bmatrix} 5/3 \\ 3/2 \end{bmatrix}$$

The three column vectors on the right of this equation thus represent, not only

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix};$$

they also represent

$$\begin{bmatrix} y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} z_{30} \\ z_{40} \end{bmatrix}, \quad \begin{bmatrix} z_{31} \\ z_{41} \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} z_{32} \\ z_{42} \end{bmatrix}.$$

In other words, given a unit matrix of coefficients for the basic variables, the first stage (except for the bottom row) of the simplex tableau can be written down directly from the coefficients of the constraints and the functional.

Following this procedure, we could have immediately written down the last stage of Table I as our initial stage, and have seen from the signs of z_{01} and z_{02} that our initial stage also gave us the optimum solution. On the other hand, had this not been the optimum solution, we could readily have calculated the z's for successive stages from algorithms.

APPENDIX TO CHAPTER III

DERIVATION OF ALGORITHMS FOR CALCULATING SUCCESSIVE Z'S

From equation (1a):

$$\overset{m}{\underset{u=1}{y_u}} = z_{u0} - \sum_{k=1}^{n-m} z_{u,m+k} y_{m+k}$$

If y_r is the particular y_u to be removed, and if y_{m+h} is the particular y_{m+k} to be introduced, at the next stage,

$$y_r + \sum_{k=1}^{n-m} z_{r,m+k} y_{m+k} = z_{r0}$$

Since y_{m+h} is a particular y_{m+k}, it is included in the second term on the left of this last equation, and we can divide this term into two parts to get:

$$y_r + z_{r,m+h} y_{m+h} + \sum_{k=1 \neq h}^{n-m} z_{r,m+k} y_{m+k} = z_{r0}$$

Dividing this last equation through by $z_{r,m+h}$, we have:

$$(3) \qquad y_{m+h} = \frac{z_{r0}}{z_{r,m+h}} - \left[\frac{1}{z_{r,m+h}} y_r + \sum_{k=1 \neq h}^{n-m} \frac{z_{r,m+k}}{z_{r,m+h}} y_{m+k} \right], \; z_{r,m+h} > 0$$

Similarly, we can also write (1a) in the form:

$$\overset{m}{\underset{u=1}{y_u}} + z_{u,m+h} y_{m+h} + \sum_{k=1 \neq h}^{n-m} z_{u,m+k} y_{m+k} = z_{u0}$$

If we now substitute for y_{m+h} in this last equation, the lengthy expression on the right of equation (3), we get:

$$
\begin{aligned}
(4)\qquad \overset{m}{\underset{u=1\neq h}{y_u}} &= z_{u0} - \frac{z_{u,m+h}}{z_{r,m+h}} z_{r0} \\
&\quad - \sum_{k=1\neq h}^{n-m} \left[z_{u,m+k} - \frac{z_{u,m+h}}{z_{r,m+h}} z_{r,m+k} \right] y_{m+k} + \frac{z_{u,m+h}}{z_{r,m+h}} y_r
\end{aligned}
$$

$z_{m+h,0}$, $z_{m+h,m+k}$, *and* $z_{m+h,r}$:

Since y_r and all

$$\overset{n-m}{\underset{k=1\neq h}{y_{m+k}}}$$

are zero, equation (3) gives us algorithm:

$$\text{(a)}\qquad y_{m+h} = z_{m+h,0} = \frac{z_{r0}}{z_{r,m+h}}$$

Equation (3) also provides us with two other algorithms, for (3) is really a statement of (1a) with particular reference to y_{m+h}. Consequently, just as the first term on the right of equation (3) is $z_{m+h,0}$, so do the coefficients in (3) of y_r and y_{m+k} represent $z_{m+h,r}$ and $z_{m+h,m+k}$, respectively. Therefore we have algorithms:

$$\text{(b)}\qquad z_{m+h,r} = \frac{1}{z_{r,m+h}}$$

$$\text{(c)}\qquad z_{m+h,m+k} = \frac{z_{r,m+k}}{z_{r,m+h}},\ k \neq h$$

Computing the z's for the entire y_{m+h} row in the successive stages of the tableau is, therefore, reduced to a simple set of routine calculations.

z_{u0}', $z_{u,m+k}'$, *and* z_{ur}:

Since y_r and all

$$\overset{n-m}{\underset{k=1\neq h}{y_{m+k}}}$$

are zero, equation (4) gives us an algorithm for all new

$$\overset{m}{\underset{u=1\neq r}{y_u'}}$$

$$\text{(d)}\qquad y_u' = z_{u0}' = z_{u0} - \frac{z_{u,m+h}}{z_{r,m+h}} z_{r0}$$

Equation (4) also provides us with two other algorithms, in much the same way as did (3). For, just as the first two terms on the right of (4) give us $y_u' = z_{u0}'$ for all the new basic variables (except y_{m+h}), so are the lengthy coefficients of y_{m+k} and y_r the $z_{u,m+k}'$ and z_{ur}, respectively. Therefore, we have the algorithms:

$$\text{(e)} \qquad z_{u,m+k}' = z_{u,m+k} - \frac{z_{u,m+h}}{z_{r,m+h}} z_{r,m+k}$$

$$\text{(f)} \qquad z_{ur} = -\frac{z_{u,m+h}}{z_{r,m+h}}$$

Algorithms (d), (e), and (f) thus make possible a routine of calculation of all the remaining z's of the tableau, except those of the bottom row.

z_{00}', $z_{0,m+k}'$ *and* z_{0r}':

As we know, our functional to be maximized can be expressed as

$$f(y) = z_{00} + \sum_{k=1}^{n-m} z_{0,m+k} y_{m+k}$$

So

$$z_{00}' = z_{00} + z_{0,m+h} y_{m+h} + \sum_{k=1 \neq h}^{n-m} z_{0,m+k} y_{m+k}$$

If we once more substitute the right side of (3) for y_{m+h} in this last expression, we get:

$$\text{(5)} \qquad \begin{aligned} z_{00}' = z_{00} &+ \frac{z_{0,m+h}}{z_{r,m+h}} z_{r0} - \frac{z_{0,m+h}}{z_{r,m+h}} y_r \\ &+ \sum_{k=1 \neq h}^{n-m} \left[z_{0,m+k} - \frac{z_{0,m+h}}{z_{r,m+h}} z_{r,m+k} \right] y_{m+k} \end{aligned}$$

From this we can readily deduce algorithms:

$$\text{(g)} \qquad z_{00}' = z_{00} + \frac{z_{0,m+h}}{z_{r,m+h}} z_{r0}$$

$$\text{(h)} \qquad z_{0r}' = -\frac{z_{0,m+h}}{z_{r,m+h}}$$

$$\text{(i)} \qquad z_{0,m+k}' = z_{0,m+k} - \frac{z_{0,m+h}}{z_{r,m+h}} z_{r,m+k}$$

The importance of the pivot in all this is manifest: it appears in each of the nine algorithms.

CHAPTER IV

DEGENERACY

Up to this point, all of the examples chosen to illustrate the fundamentals of linear programming have behaved nicely. Not all problems exhibit such exemplary behavior, and this chapter and the next are concerned with three difficulties sometimes encountered. This chapter is devoted to the problem of degeneracy, i.e., encountering a basic feasible solution (which may, or may not, be the optimum solution) involving a zero value for one (or more) of the basic variables. Chapter V is concerned with the other two difficulties.

Degeneracy might be defined, rather loosely, as a solution in which at least one of the basic variables has a value of zero. More mathematically, degeneracy is sometimes defined as a case in which *some* submatrix of order m from the matrix

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ a_{m1} & \cdots & a_{mn} & b_m \end{bmatrix}$$

is singular (i.e., the determinant of the submatrix is zero).

A simple example may help to make this definition clear. Consider the following set of requirements equations:

$$\begin{aligned} 4y_1 + y_2 + 2y_3 + 3y_4 &= 10 \\ 3y_1 + 2y_2 + 3y_3 + 4y_4 &= 10 \\ 2y_1 + 3y_2 + 2y_3 + 3y_4 &= 10 \end{aligned}$$

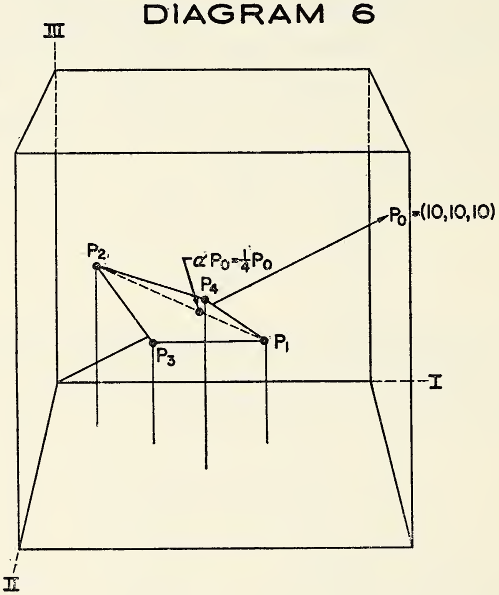
Let us now extract the coefficients of the first two columns on the left side, and the coefficients on the right side, of these equations to get the following submatrix of order three:

$$\begin{bmatrix} 4 & 1 & 10 \\ 3 & 2 & 10 \\ 2 & 3 & 10 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}$$

The rule for evaluating the determinant of a matrix of order three is: $a_{11}a_{22}b_3 - a_{11}b_2a_{32} + a_{12}b_2a_{31} - a_{12}a_{21}b_3 + b_1a_{21}a_{32} - b_1a_{22}a_{31}$. The value of our determinant is, then: $80 - 120 + 20 - 30 + 90 - 40 = 0$. Thus the above set of equations does have a submatrix of order m, the determinant of which is zero, and we must have a case of degeneracy.

We can visualize what this means from Diagram 6, which depicts this three-dimensional set of equations. Once more we find ourselves looking into the jaws of a "trap." In fact, it is the same trap as appeared in Diagram 4; but, because the right side of the third equation has been increased to 10, the P_0 ray disappears from view behind the trap, never piercing any of the plane surfaces (within the boundaries defined by the edges of the tetrahedron) in its journey from the origin to P_0. But the P_0 ray does *touch* the tetrahedron formed by $P_1P_2P_3P_4$ —at the coordinates point of (5/2, 5/2, 5/2), which point lies on the *edge* of the tetrahedron represented by the line connecting P_1 and P_2.

DIAGRAM 6



This means that, in this particular case, there is one (and only one) solution to our set of equations which will have only non-negative components for $y = (y_1, y_2, y_3, y_4)$. This unique solution is $y = (2, 2, 0, 0)$, and it is a solution involving positive values for less than m (since $m = 3$) variables.

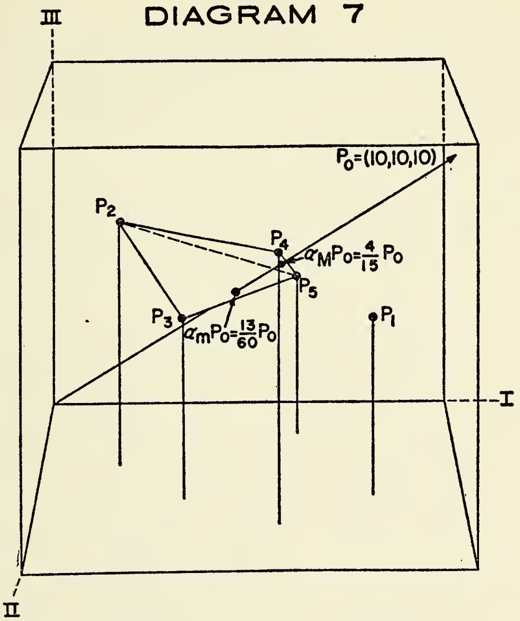
The problem just considered is misleadingly simple. The fact that one can encounter degeneracy in a problem in which not only the best solution, but even the only solution, involves less than m positive-value variables suggests that degeneracy may be less opprobrious than it sounds. While this is true, our example gives degeneracy a cleaner bill of health than it deserves.

Suppose we add a fifth variable to our three requirements equations and formulate the following problem:

$$4y_1 + y_2 + 2y_3 + 3y_4 + 3y_5 = 10$$
$$3y_1 + 2y_2 + 3y_3 + 4y_4 + y_5 = 10$$
$$2y_1 + 3y_2 + 2y_3 + 3y_4 + 2y_5 = 10$$
$$\text{Maximize } y_1 + y_2 + y_3 + y_4 + y_5$$

Since this is exactly the same problem as the preceding one, except for the addition of a fifth variable, we know that the P_0 ray will still cut the P_1P_2 line at the point $\alpha P_0 = 1/4 P_0 = (5/2, 5/2, 5/2)$. The addition of the fifth variable, *along with the inclusion of this variable in the functional to be maximized*, has, however, greatly altered the nature of the problem—as can be seen from Diagram 7.

DIAGRAM 7


III
$P_0=(10,10,10)$
P_2
P_4
$\alpha_M P_0 = \frac{4}{15} P_0$
P_5
P_3
P_1
$\alpha_m P_0 = \frac{13}{60} P_0$
I
II

In Diagram 7, the points P_2, P_3, P_4, and P_5 have been joined (except for the P_3P_4 edge of the tetrahedron) to form a new trap. P_1 has been left in solitary grandeur, but if we joined P_1 and P_2, it would cut the P_0 ray about midway between $\alpha_m P_0$ and $\alpha_M P_0$. In other words, *every* basic feasible solution involving y_1 and y_2 would result in $y_1 = y_2 = 2$ and a zero value for whatever one of the other three variables was chosen as the third basic variable in the search for the optimum solution (and, of course, the two non-basic variables would also have zero values).

More important, for our purposes, the segment of this imaginary line joining P_1 and P_2 which lies between P_2 and the point at which the P_1P_2 line would cut the $P_3P_4P_5$ plane lies entirely inside the two jaws of our trap. This is, of course, necessarily so, since $\alpha P_0 = 1/4P_0$ lies between $\alpha_m P_0$ and $\alpha_M P_0$. The $y_1 = y_2 = 2$ solution cannot, therefore, be the solution which maximizes

$$\sum_{j=1}^{5} y_j.$$

Parenthetically, however, we can readily see here why the nature of the functional to be maximized determines whether α_m will, in fact, have in its denominator those values for $y = (y_1, y_2, \cdots, y_5)$ which will maximize the functional.

If our problem were simply to maximize $y_1 + y_2 + y_3$, α_m [which actually has as its denominator $y = (0, 10/13, 30/13, 0, 20/13)$] would imply a functional value of 40/13. This 40/13 is smaller than the functional value for a solution in which $y = (2, 2, 0, 0, 0)$, for this latter solution means a functional value of $2 + 2 + 0 = 4$, no matter which of the last three variables is taken as the third basic variable.

Now we know from Diagram 7 that, with a functional requiring maximization of $y_1 + y_2 + y_3 + y_4 + y_5$, the optimum solution involves y_2, y_3, and y_5 as the basic variables, and we know that this problem carries also the possibility of encountering degeneracy. Let us, therefore, choose an initial attack on this problem which will assure us of meeting degeneracy before we reach the optimum solution. In this way, we can understand more clearly the manner in which degeneracy is likely to constitute a difficulty, why, and what can be done about it.

We choose as our initial solution:

$$\begin{array}{l} c_2 = 1 \\ c_4 = 1 \\ c_5 = 1 \end{array} \quad \begin{bmatrix} y_2 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} -5/16 & -3/16 & 9/16 \\ 1/16 & 7/16 & -5/16 \\ 3/8 & -3/8 & 1/8 \end{bmatrix}$$

$$\cdot \left\{ \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix} - \overset{c_1 = 1}{\begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}} y_1 - \overset{c_3 = 1}{\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}} y_3 \right\}$$

$$= \begin{bmatrix} 5/8 \\ 15/8 \\ 5/4 \end{bmatrix} - \begin{bmatrix} -\ 11/16 \\ 15/16^* \\ 5/8^* \end{bmatrix} y_1 - \begin{bmatrix} -\ 1/16 \\ 13/16 \\ -\ 1/8 \end{bmatrix} y_3$$

$$z_{00} = 15/4 \qquad z_{01} = 1/8 \qquad z_{03} = 3/8$$

With both z_{01} and z_{03} positive, we know that we can improve z_{00} by introducing either y_1 or y_3 as basic variables. If we choose to introduce y_3, we should remove y_4, and this would take us immediately to the optimum solution, so this course has no educational value for us. We elect, therefore, to introduce y_1 as the new basic variable.

Having chosen y_1 as the new variable to be introduced into the basis, which variable should be removed from the basis? We know the rule tells us to remove the variable for which the positive ratio $z_{u0}/z_{u,m+h}$ is the smallest. But we have here a situation we have not previously encountered: a tie, since $15/8 \div 15/16$ and $5/4 \div 5/8$ are both ratios of 2:1. Each of these alternatives as the second-stage solution results in one basic variable of zero value, as can be seen from the solutions under *A* and *B*.

A:

$$\begin{array}{l} c_2 = 1 \\ c_1 = 1 \\ c_5 = 1 \end{array} \quad \begin{bmatrix} y_2 \\ y_1 \\ y_5 \end{bmatrix} = \begin{bmatrix} -\ 4/15 & 2/15 & 1/3 \\ 1/15 & 7/15 & -\ 1/3 \\ 1/3 & -\ 2/3 & 1/3 \end{bmatrix}$$

$$c_4 = 1 \qquad c_3 = 1$$

$$\cdot \left\{ \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix} y_4 - \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} y_3 \right\}$$

$$= \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 11/15 \\ 16/15 \\ -\ 2/3 \end{bmatrix} y_4 - \begin{bmatrix} 8/15 \\ 13/15^* \\ -\ 2/3 \end{bmatrix} y_3$$

$$z_{00} = 4 \quad z_{04} = -\ 2/15 \qquad z_{03} = 4/15$$

B:

$$\begin{array}{l} c_2 = 1 \\ c_4 = 1 \\ c_1 = 1 \end{array} \quad \begin{bmatrix} y_2 \\ y_4 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1/10 & -\ 3/5 & 7/10 \\ -\ 1/2 & 1 & -\ 1/2 \\ 3/5 & -\ 3/5 & 1/5 \end{bmatrix}$$

$$c_5 = 1 \qquad c_3 = 1$$

$$\cdot\left\{\begin{bmatrix}10\\10\\10\end{bmatrix} - \begin{bmatrix}3\\1\\2\end{bmatrix}y_5 - \begin{bmatrix}2\\3\\2\end{bmatrix}y_3\right\}$$

$$= \begin{bmatrix}2\\0\\2\end{bmatrix} - \begin{bmatrix}11/10\\-3/2\\8/5\end{bmatrix}y_5 - \begin{bmatrix}-1/5\\1^*\\-1/5\end{bmatrix}y_3$$

$$z_{00} = 4 \quad z_{05} = -1/5 \qquad z_{03} = 2/5$$

With alternative A as the second-stage solution, despite the zero value for y_5, our pivot is z_{13}. This means we should substitute y_3 for y_1 as a basic variable in the third-stage solution. As we know, this will result in the optimum solution in the third stage; so degeneracy offers no obstacle to finding the optimum solution, if we choose alternative A as the second-stage solution.

With alternative B as the second-stage solution, our pivot is z_{43}. This means we should substitute y_3 for y_4 as a basic variable in the third-stage solution. But this, as we know, will result in unchanged values for $y_1 = y_2 = 2$ and a zero value for y_3 in the third-stage solution. How can we be sure that our third-stage solution will not result in a z_{34} pivot, that our fourth-stage solution will not again result in a z_{43} pivot, and so on, with an endless bouncing back and forth between y_3 and y_4? Or how can we be sure that our third-stage solution will not result in a z_{35} pivot (which would result, in the fourth-stage solution, in substituting y_5 for y_3 as the basic variable with zero value), that the fourth-stage solution will not result in a z_{54} pivot, that the fifth-stage solution will not result in a z_{43} pivot, and so on through an endless cycle of substituting y_5 for y_3, y_4 for y_5, y_3 for y_4, and y_5 for y_3 again, ad infinitum?

Actually, if we do choose alternative B, we still get to the optimum solution. The third-stage solution results in a z_{15} pivot; so, by substituting y_5 for y_1 in the basis, we reach the optimum solution in the fourth stage—only one stage later than if we had chosen alternative A. Indeed, the probability of running into a problem which will lead to a fruitless cycling process such as we have just considered as a possibility, is supposed to be very small. Nevertheless, encountering the cycling process is at least a possibility; this is where degeneracy causes real trouble, and it is clearly sensible to do what one can to protect oneself against such a contingency, remote though the contingency may be.

A method for coping with this contingency is given by A. Charnes.[1] To understand the rationale for this method, let us begin by considering where our (potential) troubles began. For, although our fears proved groundless, we did get an unmistakable signal of at least the possibility of our encountering a cycling process

This signal came at the very outset, when we had a choice between y_1 and y_3 as the variable to be *introduced* into the basis at the second stage, and we made the poorer choice of y_1. Parenthetically, although there is no certain rule for making the best choice from several variables with $z_{0,m+k} > 0$, our choice was less than brilliant, since the signal of danger was staring us in the face from the z_{01} column. But our poor choice can be justified on educational grounds, as well as on the grounds that the situation might have been one in which we had no choice but y_1 (i.e., z_{03} might have been negative).

Our signal came in the form of the two identical positive ratios z_{40}/z_{41} and z_{50}/z_{51}. At this point, our otherwise infallible rule for choosing the variable to be *removed* from the basis failed us. An inevitable consequence of two such identical ratios is that *both* of the basic variables associated with these ratios will become zero in the next stage—one of them by assumption, and the other by virtue of the inexorable laws of mathematics.

Suppose, for example, that we make z_{41} our pivot. Then, by assumption, y_4 becomes zero in the next stage, and

$$y_1 = z_{10} = \frac{z_{40}}{z_{41}}$$

in the next stage. What happens to y_5' (i.e., to y_5 in the next stage)? We know that $y_5' = z_{50}'$, and we know the rule is that

$$z_{u0}' = z_{u0} - \frac{z_{u,m+h}}{z_{r,m+h}} z_{r0}.$$

But what is an alternative expression for z_{u0}? If we have two identical ratios, then

$$\frac{z_{u0}}{z_{u,m+h}} = \frac{z_{r0}}{z_{r,m+h}}$$

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[1] A. Charnes, "Optimality and Degeneracy in Linear Programming," *Econometrica*, 1952, pp. 160–172.

is a general statement of these two ratios, and we can thus say that

$$z_{u0} = \frac{z_{u,m+h}}{z_{r,m+h}} z_{r0}.$$

Substitution of this alternative expression for z_{u0} into

$$z_{u0}' = z_{u0} - \frac{z_{u,m+h}}{z_{r,m+h}} z_{r0}$$

thus inevitably means that $z_{u0}'=0$. This is, of course, just another way of saying that the appearance of such identical ratios is no accident, but can happen only when degeneracy is present.

This being so, and knowing that degeneracy means a P_0 ray cutting an edge, rather than passing through plane surfaces, of some tetrahedron (where we are dealing with a three-dimensional requirements space), what is needed is some method of temporarily displacing the P_0 ray towards the interior of the tetrahedron, then replacing the P_0 ray once the final solution is reached. Diagrammatically, we want a method which will displace the P_0 ray in Diagram 6 (or, though less easily visualized, in Diagram 7) so that it cuts the $P_1P_2P_3$ and $P_1P_2P_4$ planes, instead of merely cutting the P_1P_2 edge of the relevant tetrahedron. At the same time, since one really does not know in advance whether one is dealing with a problem involving degeneracy, we want a method which will not throw us into an artificial problem involving degeneracy. Charnes' method is such.

Charnes' method is to add carefully defined polynomials to each of the

$$\overset{m}{\underset{u=1}{z_{u0}}} = y_u.$$

To grasp the meaning of the polynomials' definition, consider any single row u $(u=1, \cdots, m)$, say the first row so that $u=1$, of the extended simplex tableau. Perhaps it would also be helpful to assume more specifically that (a) there are three basic variables, so that $m=3$, and (b) there are two non-basic variables, so that $n=5$. Expressing this row both generally and specifically, as it would appear in the tableau, we have:

		y_1	y_2	$y_m = y_3$	$y_{m+1} = y_4$	$y_n = y_5$
$y_u = y_1$	$z_{u0} = z_{10}$	$z_{uu} = z_{11} = 1$	0	0	$z_{u,m+1} = z_{14}$	$z_{un} = z_{15}$

The definition of the polynomial to be added to each of the

$$\overset{m}{\underset{u=1}{z_{u0}}}$$

is:

$$\overset{m}{\underset{u=1}{T_u}} \equiv z_{uu}E^u + z_{u,m+1}E^{m+1} + \cdots + z_{un}E^n,$$

where E is a very small positive number. The polynomial is thus composed of $(1+n-m)$ terms: one for the basic variable on the far left of the row, and one term for each of the $(n-m)$ non-basic variables. The coefficients of these terms are the $(1+n-m)$

$$\overset{5}{\underset{n=1}{z_{un}}}$$

in the row (z_{uu} is, of course, always unity, while there are always zero coefficients in that particular row under the remaining basic variables); and the power to which the E associated with each of these z's is raised is the same as the suffix of the variable heading that z's column.

To express these polynomials with specific reference to the case of the above three basic variables and two non-basic variables:

$$T_1 = E + z_{14}E^4 + z_{15}E^5$$
$$T_2 = E^2 + z_{24}E^4 + z_{25}E^5$$
$$T_3 = E^3 + z_{34}E^4 + z_{35}E^5$$

With the polynomials to be added to z_{u0} defined in this manner, it is impossible for any two ratios, say

$$\frac{z_{10} + T_1}{z_{1,m+1}} \text{ and } \frac{z_{20} + T_2}{z_{2,m+1}},$$

to be equal, and we can thus reach the optimum solution without incurring the risk of getting into a cycling process. Then, having reached the optimum solution, the solution can be properly evaluated be setting $E=0$.

This follows because either

$$\frac{z_{10}}{z_{1,m+1}} \text{ and } \frac{z_{20}}{z_{2,m+1}}$$

are equal ratios, or they are not equal ratios. If they are not equal ratios, addition of T_1 and T_2, respectively, to their numerators will not throw us into an "artificial" equality of the ratios, because E can be chosen small enough to avoid any such effect. If the original ratios are equal, however, inclusion of the appropriate polynomial in the original ratios' respective numerators will result in the relative size of the artificial ratios depending on the lowest-power E to induce inequality—this, because the higher powers of a very small number will be relatively unimportant. A unique choice as to the variable to be removed from the basis is thus assured at every stage.

To illustrate this, consider once more the first-stage solution to our initial degeneracy problem in five variables. Adding to that solution columns for the three basic variables, in order to have all the relevant portions of an extended simplex tableau before us, we have:

		y_2	y_4	y_5	y_1	y_3
y_2	$z_{20} = 5/8$	$z_{22} = 1$			$z_{21} = -11/16$	$z_{23} = -1/16$
y_4	$z_{40} = 15/8$		$z_{44} = 1$		$z_{41} = 15/16$	$z_{43} = 13/16$
y_5	$z_{50} = 5/4$			$z_{55} = 1$	$z_{51} = 5/8^*$	$z_{53} = -1/8$

Since we chose y_1 as the new variable to be introduced into the basis in the second stage, we were confronted with the two equal ratios of

$$\frac{z_{40}}{z_{41}} \text{ and } \frac{z_{50}}{z_{51}} \cdot$$

If we had added the above-defined polynomials to the z_{u0}, the *numerators* of the three ratios would have been:

$$\begin{array}{lllll} y_2 & 5/8 - 11/16E & + E^2 & - 1/16E^3 & \\ y_4 & 15/8 + 15/16E & & + 13/16E^3 & + E^4 \\ y_5 & 5/4 + 5/8E & & - 1/8E^3 & + E^5 \end{array}$$

The first E term leaves the two pertinent ratios unchanged, but the E_3 term makes

$$\frac{z_{50}}{z_{51}} < \frac{z_{40}}{z_{41}},$$

so we replace y_5 with y_1 in the basis.

This is the beginning of the alternative B course we have previously considered, but inclusion of the polynomials will alter our third-stage solution significantly, even though slightly. The values for y_2, y_4, y_1 (the new basic variables) now become:

$$\begin{array}{ll}
y_2 & 2 + E^2 - 1/5E^3 + 11/10E^5 \\
y_4 & 0 + E^3 + E^4 - 3/2E^5 \\
y_1 & 2 + E - 1/5E^3 + 8/5E^5
\end{array}$$

We encountered no equal-ratio problem at this stage, but got a clear mandate to replace y_4 by y_3, and this mandate still holds. Originally, however, we had $y_4 = 0$ in this stage, and now we have $y_4 > 0$.

We might note, parenthetically, that, had the subscripts of y_3 and y_5 been reversed, we should have had $y_4 < 0$ (and negative values for y components are not permissible). This would follow because y_3 (formerly y_5) has the negative coefficient, $-3/2$, and would now also be associated with the lowest-power E (E^3, rather than E^5). But this contingency can always be avoided by simply re-naming the basic variables. Thus y_2, y_4, and y_1 could have been renamed y_1, y_2, and y_3, respectively. The lowest power term in the old y_4 row would then be $z_{22}E^2 = E^2$, and this would dwarf all other higher-power terms of E.

Skipping over the third-stage solution to the final solution, inclusion of the polynomials results in the following values for y_5, y_2, and y_3:

$$\begin{array}{lll}
c_5 = 1 & y_5 & 20/13 + 10/13E + 2/13E^4 + E^5 \\
c_2 = 1 & y_2 & 10/13 - 8/13E + E^2 + 1/13E^4 \\
c_3 = 1 & y_3 & 30/13 + 15/13E + E^3 + 16/13E^4 \\
 & z_{00} = & 60/13 + 17/13E + E^2 + E^3 + 19/13E^4 + E^5
\end{array}$$

Knowing from the signs of z_{04} and z_{01} that this is the final solution, we set $E = 0$ to obtain the true $z_{00} = 60/13$.

Inclusion of the polynomials in the successive stages of the solution thus does two things: (1) it avoids the possibility of encountering equal ratios, and what is traceable to the same phenomenon of degeneracy, (2) it avoids the appearance of a basic variable of zero value—unless, of course, the *final* solution actually should have a basic variable of zero value.

CHAPTER V

OTHER DIFFICULTIES

In addition to the problem of degeneracy, one may also encounter: (a) problems for which no finite maximum (minimum) exists; (b) problems in which it is not easy to find a basic feasible solution with which to begin the iterative process—and, indeed, it may even be that no such solution exists. How to cope with these two complications is the subject of this chapter.

NO FINITE MAXIMUM (MINIMUM)

Consider the following problem:

$$\begin{aligned} -2y_1 + y_2 + y_3 \qquad\qquad &= 2 \\ y_1 - 2y_2 \qquad + y_4 \qquad &= 2 \\ y_1 + y_2 \qquad\qquad - y_5 &= 5 \end{aligned}$$

$$\text{Maximize } y_1 + y_2 + y_3 + y_4$$

This problem has no finite maximum. In fact, we could also use these same requirements equations to illustrate a case with no finite minimum, if we changed the functional to: minimize $y_2 - y_1$. Since it is easier to visualize the lack of finite maximum, however, we shall consider the maximization problem.

In the problem outlined in the simplex tableau of Chapter III, we encountered one of only two types of situations. In the first place, we had the type in which for *some* k, $z_{0,m+k} > 0$, and for *some* u, $z_{u,m+k} > 0$. In the first stage, for example, we found that $z_{03} = 4/5$, and $z_{13} = 9/10$. And, from this, we deduced that y_3 could profitably be substituted for y_1 as a basic variable. In brief, this type of situation with respect to $z_{0,m+k}$ and $z_{u,m+k}$ means the solution can be bettered, and tells us where.

Secondly, we encountered the type of situation in which for *all* k, $z_{0,m+k} \leq 0$. This type we did not encounter until the final stage, with $z_{01} = -7/6$ and $z_{02} = -5/12$. This type of situation with respect to the $z_{0,m+k}$ we learned meant that the optimum (maximum) solution had been reached.

There exists a third type of situation; this and the other two represent the three mutually exclusive and collectively exhaustive possibilities. This is the type in which for *some* k, $z_{0,m+k} > 0$, and for *all*

u, $z_{u,m+k} \leq 0$. We are about to encounter this type of situation, and it means we are confronted with a linear functional whose "maximum" is infinity.[1]

To illustrate this, suppose we begin by solving for $y=(y_1, y_2, y_3)$, also calculating the $z_{u,m+k}$ and $z_{0,m+k}$ as we do so. In doing this, we find that (for convenience, the coefficients from the functional are listed on the left of the basic variables and above the non-basic variables):

$$\begin{array}{l} c_1 = 1 \\ c_2 = 1 \\ c_3 = 1 \end{array} \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 2/3 \\ 0 & -1/3 & 1/3 \\ 1 & 1 & 1 \end{bmatrix} \left\{ \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix} - \overset{c_4 = 1}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} y_4} - \overset{c_5 = 0}{\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} y_5} \right\}$$

$$= \begin{bmatrix} 4 \\ 1 \\ 9 \end{bmatrix} - \begin{bmatrix} 1/3 \\ -1/3 \\ 1 \end{bmatrix} y_4 - \begin{bmatrix} -2/3 \\ -1/3 \\ -1 \end{bmatrix} y_5$$

$$z_{00} = 14 \qquad z_{04} = 0 \qquad z_{05} = 2$$

With $z_{04}=0$, we conclude that substitution of y_4 for y_3 as a basic variable will neither help nor hurt our cause. And this is precisely right, for this substitution would result in $y=(y_1, y_2, y_4)=(1, 4, 9)$, still leaving $z_{00}=14$.

More important for our problem, with $z_{05}=2>0$, we should think we could improve our functional by substituting y_5 for one of the present basic variables. But all

$$\overset{3}{\underset{u=1}{z_{u5}}}$$

are negative, so our rule for selecting the variable to be removed from the basis appears to break down. In fact, however, none of the variables *should* be removed, although y_5 should indeed be added. In brief, the "optimum" solution involves a "solution" in $(m+1)$ variables.

---

[1] If the functional were to be minimized, the inequalities involving the $z_{0,m+k}$ should, of course, have their signs reversed for all three types of situation. Parenthetically, some writers (e.g., S. Vajda, *The Theory of Games and Linear Programming* [London: Methuen & Co. Ltd., 1956]) define

$$z_{0,m+k} = \sum_{u=1}^{m} c_u z_{u,m+k} - c_{m+k}.$$

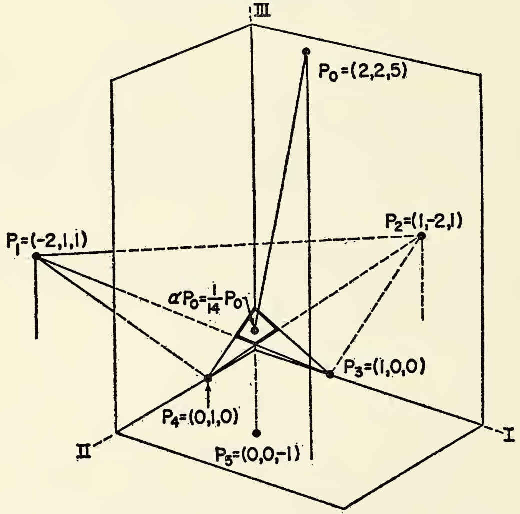
This reverses the sign signals indicating an optimum solution: i.e., for *all* k, $z_{0,m+k} \leq 0$ would signify the optimum *min*imum had been reached, rather than the optimum *max*imum.

To state the matter differently, the following table of solutions (all of which satisfy the requirements equations) could obviously be expanded indefinitely, so that the linear functional to be maximized tends towards infinity:

y_1	y_2	y_3	y_5
4	1	9	0
6	2	12	3
8	3	15	6

The reason for this can be easily visualized in Diagram 8. A little intuitive reflection makes it clear that the two triangular planes $P_1P_2P_3$ and $P_1P_2P_4$ are bounded by points, all four of which (i.e., P_1, P_2, P_3, and P_4) lie in the same plane, and hence cut into the all-positive section (the *interior* of the box) in our diagram in such a way as to have the common plane section represented by the small heavily lined quadrilateral. Through this quadrilateral plane, the ray from the origin to P_0 passes. At this point, we have $\alpha = 1/14$, whether the denominator is the sum of the positive values for $y = (y_1, y_2, y_3)$ or whether it is the sum of the positive values for $y = (y_1, y_2, y_4)$.

DIAGRAM 8



But y_3, y_4, and y_5 each appear only once in different equations, y_5 has a negative coefficient (i.e., -1), and the size and signs of the y_1 and y_2 coefficients are such that we can assign increasingly greater values to y_5, with a resultant increase also in each of y_1, y_2, and y_3 (or of y_1, y_2, and y_4). We can thus view successive increases in y_5 as the equivalent of making b_3 (which equals 5, when $y_5=0$) larger and larger. Thus the *height* of P_0 can be expanded indefinitely, since b_3 is plotted as the height of P_0. While this also raises the *absolute* height at which the P_0 ray cuts the quadrilateral plane, this latter height gets smaller and smaller, *relative* to the height of P_0, so that α gets smaller. Hence the P_0 ray approaches the vertical axis rising from the origin as the height of P_0 approaches infinity. This demonstrates geometrically what we already know algebraically to be so: since these successively smaller α's have a denominator which is the sum of either y_1, y_2, and y_3 or y_1, y_2, and y_4, there is no finite maximum to our functional.

In addition to the criterion of $z_{0,m+k}>0$ for *some k* and $z_{u,m+k}\leq 0$ for *all u*, as a means of recognizing a problem with no finite maximum, there is a second method one can employ. This involves the introduction of an artificial constraint with a single artificial variable, y_{n+1}, making this constraint:

$$y_1 + \cdots + y_n + y_{n+1} = \beta,$$

where $\beta=$a large but finite number. This means that, if a true optimum solution exists, it will be accompanied by a value for our artificial variable of:

$$y_{n+1} = \beta - (y_1 + \cdots + y_n).$$

The following is such a problem, with y_4 being the artificial variable in the artificial constraint:

$$\begin{aligned}
2y_1 + 3y_2 \qquad\qquad\quad &= 6\\
4y_1 + y_2 + 11y_3 \qquad &= 5\\
y_1 + y_2 + y_3 + y_4 &= \beta\\
\text{Maximize } y_1 + y_2 + y_3 &
\end{aligned}$$

Including our artificial constraint, we have a problem involving a 3×3 inverse matrix, rather than a 2×2 inverse matrix. The final stage of the solution is:

$$\begin{array}{l} c_1 = 1 \\ c_2 = 1 \\ c_4 = 0 \end{array} \quad \begin{bmatrix} y_1 \\ y_2 \\ y_4 \end{bmatrix} = \begin{bmatrix} 2/5 & -1/5 & 0 \\ -1/10 & 3/10 & 0 \\ -3/10 & -1/10 & 1 \end{bmatrix} \left\{ \begin{bmatrix} 6 \\ 5 \\ \beta \end{bmatrix} - \overset{c_3 = 1}{\begin{bmatrix} 0 \\ 11 \\ 1 \end{bmatrix}} y_3 \right\}$$

$$= \begin{bmatrix} 7/5 \\ 9/10 \\ \beta - 23/10 \end{bmatrix} - \begin{bmatrix} -11/5 \\ 33/10 \\ -1/10 \end{bmatrix} y_3$$

$$z_{00} = 23/10 \qquad z_{03} = -1/10$$

With $z_{03} < 0$, we cannot improve z_{00} by introducing y_3 as a basic variable, and we have reached a finite maximum of 23/10 with the artificial variable, $y_4 = \beta - 23/10$, as it should do.

On the other hand, if we introduce an artificial variable, y_6, into an artificial constraint of

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = \beta,$$

making this a fourth constraint in the *initial* problem of this section, β will appear in the final values for

$$\overset{5}{\underset{j=1}{y_j}}.$$

The final stage of the solution to this problem now becomes:

$$\begin{array}{l} c_1 = 1 \\ c_2 = 1 \\ c_3 = 1 \\ c_5 = 0 \end{array} \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_5 \end{bmatrix} = \begin{bmatrix} -2/9 & 1/9 & 2/9 & 2/9 \\ -1/9 & -4/9 & 1/9 & 1/9 \\ 2/3 & 2/3 & 1/3 & 1/3 \\ -1/3 & -1/3 & -2/3 & 1/3 \end{bmatrix}$$

$$\cdot \left\{ \begin{bmatrix} 2 \\ 2 \\ 5 \\ \beta \end{bmatrix} - \overset{c_4 = 1}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}} y_4 - \overset{c_6 = 0}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}} y_6 \right\}$$

$$= \begin{bmatrix} 8/9 + 2/9\beta \\ -5/9 + 1/9\beta \\ 13/3 + 1/3\beta \\ -14/3 + 1/3\beta \end{bmatrix} - \begin{bmatrix} 1/3 \\ -1/3 \\ 1 \\ 0 \end{bmatrix} y_4 - \begin{bmatrix} 2/9 \\ 1/9 \\ 1/3 \\ 1/3 \end{bmatrix} y_6$$

$$z_{00} = 14/3 + 2/3\beta \qquad z_{04} = 0 \qquad z_{06} = -1/3$$

β now appears in the expression for each of the four variables y_1, y_2, y_3, and y_5, which tells us there is no finite solution to the problem. Indeed, our feasible solution column can be viewed as four equations in the five unknowns y_1, y_2, y_3, y_5, and β. Since $y_5 = 1/3\beta - 14/3$, when $y_5 = 0$, $\beta = 14$, and each of the

$$\overset{3}{\underset{j=1}{y_j}} > 0.$$

Therefore, any increase in β (over $\beta = 14$) necessarily means an improved feasible solution involving an increase in each of the four variables y_1, y_2, y_3, and y_5.

DISCOVERING AN INITIAL BASIC FEASIBLE SOLUTION

All the problems we have considered so far have not only had at least one basic feasible solution, but we have had no difficulty in discovering a basic feasible solution as our initial point of attack on the maximization (minimization) of the functional. A problem may, however, have no basic feasible solution, and, even if it has, it may be far from obvious which of numerous possibilities is such a solution. It would be helpful, therefore, to have a method which assured us of (a) an initial point of attack in the form of a basic feasible solution and (b) confirmation of the fact that no basic feasible solution exists, if no such solution does exist.

There is such a method, and to understand it let us begin by considering a problem which has no basic feasible solution:

$$\begin{array}{ll} (1) & 2y_1 + 3y_2 + 4y_3 \phantom{{}+ 2y_4} = 6 \\ (2) & 4y_1 - 3y_2 \phantom{{}+ 4y_3} + 2y_4 = -7 \end{array}$$

$$\text{Maximize } y_1 - y_2$$

Now divide (1) by 4 (i.e., by a_{13}) and, after multiplying (2) by -1 (to make b_2 positive), divide (2) by 2 (i.e., by a_{24}). This gives us:

$$\begin{array}{ll} (1) & 1/2y_1 + 3/4y_2 + y_3 \phantom{{}- y_4} = 3/2 \\ (2)' & -2y_1 + 3/2y_2 \phantom{{}+ y_3} - y_4 = 7/2 \end{array}$$

$$\text{Maximize } y_1 - y_2$$

As we can readily see from Diagram 9, this problem has no basic feasible solution, since the P_0 ray will not intersect any of the lines joining P_1, P_2, P_3, and P_4. But we assume our problem is too complicated for us to discover this by such a simple expedient.

On the other hand, if y_4 in (2)′ had been positive (so that it lay at P_a in Diagram 9), the problem would have three basic feasible solutions and we should also have an obvious starting point, namely, to take y_3 and y_4 as our initial basic variables. This suggests a method for getting started on a problem which has no obvious starting point.

In (1)′ and (2)′ we have m (in this case, of course, $m=2$) equations with all b_i non-negative. We also have $k \leq m$ variables appearing in only one equation and having a positive coefficient; in our case, $k=1$, since only y_3 meets both of these specifications—and we might have had no such variables, in which event k would have been zero.

Now suppose we make the y_j, appearing in only one equation and having positive coefficients, equal to b_i/a_{ij} and make all other variables zero. In our example, this amounts to making $y_1=y_2=y_4=0$, with y_3 having the value of 3/2. This means that k equations will be true [(1)′, in our case] and $(m-k)$ equations will not be true [(2)′, in our case]. So we add $(m-k)$ artificial non-negative variables, one per equation, to the left side of the $(m-k)$ equations not true.

Our problem now becomes:

$$(1)'' \qquad 1/2y_1 + 3/4y_2 + y_3 \qquad = 3/2$$

$$(2)'' \qquad -2y_1 + 3/2y_2 \qquad - y_4 + y_a = 7/2$$

$$\text{Maximize } y_1 - y_2$$

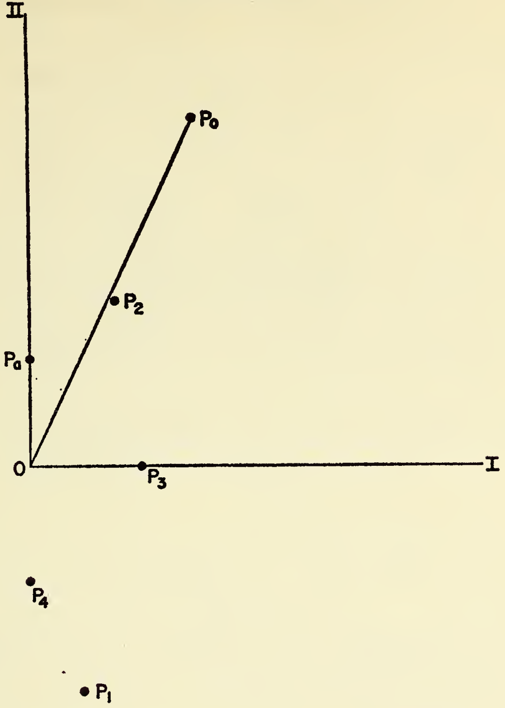
This provides us with a starting point, using y_3 and y_a as basic variables. In fact, we also know from Diagram 9 that there are three solutions to the problem involving y_a. Still we obviously do not want an artificial variable in our final solution, but need it (or them) only to get started towards the final solution.

A sure way to avoid having artificial variables in the final solution is also to introduce them into the functional, giving them coefficients $-M$ ($+M$, if the functional is to be minimized) large enough to dominate any numbers to which $-M$ will be compared. In our problem, we know from (2)″ that $y_a = 7/2 + 2y_1 - 3/2y_2 + y_4$, so introducing $-My_a$ into the functional results in a functional of

$$\text{Maximize } y_1 - y_2 - M(7/2 + 2y_1 - 3/2y_2 + y_4)$$
$$= (1-2M)y_1 - (1-3/2M)y_2 - My_4 - 7/2M$$

If a feasible solution to the original problem exists, we are now sure to discover it. This is so because the functional in a form which includes $-My_a$ cannot result in a solution which both includes y_a in

DIAGRAM 9



that solution and also maximizes the functional. If $-M$ is a large enough coefficient to dominate all others to which it is compared, then $-My_a$ is sure to do more harm towards maximizing the functional than is *any true* variable which may enter into a feasible solution. In other words, by including $-My_a$ in the functional, the definition of $-M$ makes it certain that the simplex method rules will result in a signal to remove y_a as a basic variable—if a variable exists which will be signaled into the basis. Conversely, then, if we reach a final solution which retains y_a, we can be sure no feasible solution exists.

To illustrate this no-feasible-solution case with our present problem, the first stage solution is:

$$\begin{array}{l} c_3 = 0 \\ c_a = -M \end{array} \quad \begin{bmatrix} y_3 \\ y_a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{array}{cccc} & c_1 = 1 & c_2 = -1 & c_4 = 0 \\ \left\{\begin{bmatrix} 3/2 \\ 7/2 \end{bmatrix}\right. - & \begin{bmatrix} 1/2^* \\ -2 \end{bmatrix} y_1 - & \begin{bmatrix} 3/4 \\ 3/2 \end{bmatrix} y_2 - & \left.\begin{bmatrix} 0 \\ -1 \end{bmatrix} y_4 \right\} \\ z_{00} = 0 & z_{01} = 1 & z_{02} = -1 & z_{04} = 0 \end{array}$$

This tells us to substitute y_1 for y_3 in the second stage, and this leads to:

$$\begin{array}{lcccccccc} & & & & c_3 = 0 & & c_2 = -1 & & c_4 = 0 \\ \begin{array}{l} c_1 = 1 \\ c_a = -M \end{array} & \begin{bmatrix} y_1 \\ y_a \end{bmatrix} = & \begin{bmatrix} 3 \\ 19/2 \end{bmatrix} & - & \begin{bmatrix} 2 \\ 4 \end{bmatrix} y_3 & - & \begin{bmatrix} 3/2 \\ -9/2 \end{bmatrix} y_2 & - & \begin{bmatrix} 0 \\ -1 \end{bmatrix} y_4 \\ & & z_{00} = 3 & & z_{03} = -2 & & z_{02} = -5/2 & & z_{04} = 0 \end{array}$$

In essence, this tells us that z_{00} cannot be increased by substituting y_2, y_3, or y_4 for y_1 or y_a. It also tells us, since y_a remains in the final solution, that no feasible solution exists. The z values at the bottom exclude the imaginary effect, of course, of multiplying the y_a row by $-M$.

Now consider the following case, which does have a basic feasible solution:

$$2y_1 + 3y_2 + 4y_3 = 6 \tag{1}$$

$$4y_1 - 3y_2 + 3y_4 = -5 \tag{2}$$

$$\text{Maximize } y_2 - y_1$$

Dividing (1) by 4, multiplying (2) by -1 and then dividing it by 3, and finally adding y_a to (2) we have:

$$1/2y_1 + 3/4y_2 + y_3 = 3/2 \tag{1$'$}$$

$$-4/3y_1 + y_2 - y_4 + y_a = 5/3 \tag{2$'$}$$

$$\text{Maximize } (1 + M)y_2 - (1 + 4/3M)y_1 - My_4 - 5/3M$$

Solving for y_3 and y_a, we get the following first stage of the tableau, with the figures we imagine multiplied by $-M$ repeated below the z values at the bottom (these latter are, of course, the $-M$ part of the artificial functional):

$$\begin{array}{ll} \begin{array}{l} c_3 = 0 \\ c_a = -M \end{array} & \begin{bmatrix} y_3 \\ y_a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ & \qquad\quad -M \end{array}$$

$$\begin{array}{ccccccc} & & c_1 = -1 & & c_2 = 1 & & c_4 = 0 \\ \left\{\begin{bmatrix} 3/2 \\ 5/3 \end{bmatrix}\right. & - & \begin{bmatrix} 1/2 \\ -4/3 \end{bmatrix} y_1 & - & \begin{bmatrix} 3/4 \\ 1^* \end{bmatrix} y_2 & - & \left.\begin{bmatrix} 0 \\ -1 \end{bmatrix} y_4\right\} \\ z_{00} = 0 & & z_{01} = -1 & & z_{02} = 1 & & z_{04} = 0 \\ 5/3 & & -4/3 & & 1 & & -1 \end{array}$$

Our pivot now tells us to introduce y_2 and remove y_a. Since this will provide us, in the second stage, with a true solution, we could forget about y_a. Nevertheless, let us include y_a among the non-basic variables in the second stage, which looks like this:

$$\begin{array}{ll} & \qquad\qquad\qquad c_1 = -1 \qquad\quad c_a = -M \qquad\quad c_4 = 0 \\ \begin{array}{l} c_3 = 0 \\ c_2 = 1 \end{array} & \begin{bmatrix} y_3 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 5/3 \end{bmatrix} - \begin{bmatrix} 3/2 \\ -4/3 \end{bmatrix} y_1 - \begin{bmatrix} -3/4 \\ 1 \end{bmatrix} y_a - \begin{bmatrix} 3/4^* \\ -1 \end{bmatrix} y_4 \\ & z_{00} = 5/3 \quad z_{01} = 1/3 \quad z_{0a} = -(M+1) \quad z_{04} = 1 \end{array}$$

With $c_a = -M$, and with $-M$ dominating all numbers to which it is to be compared (in this case, to $z_{2a} = 1$), we shall never get a signal to bring y_a back into the basis, for z_{0a} will always be negative.

If we elect to bring y_4 into the basis in the third stage, our pivot becomes z_{34}. This leads to the final solution of:

$$\begin{array}{ll} & \qquad\qquad\qquad c_1 = -1 \qquad\quad c_3 = 0 \\ \begin{array}{l} c_4 = 0 \\ c_2 = 1 \end{array} & \begin{bmatrix} y_4 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 2/3 \end{bmatrix} y_1 - \begin{bmatrix} 4/3 \\ 4/3 \end{bmatrix} y_3 \\ & z_{00} = 2 \quad z_{01} = -5/3 \quad z_{03} = -4/3 \end{array}$$

CHAPTER VI

DUALITY

The concept of duality is important, partially because it provides an additional method of solving a linear programming problem,[1] but, more importantly for us, because it also makes possible an understanding of the relationship between linear programming and game theory. While a precise and mathematically general statement of the nature of duality can be made in a page or two, an approach of this sort is likely to be more confusing than enlightening to the nonmathematician. For this reason, we shall begin by considering a specific problem in linear programming, and by considering the dual of this problem more or less simultaneously. To the extent that it is possible to do so during this process, we shall also draw certain generalizations as we go along. A complete generalization of the relationships between dual sets of equations will be reserved for a mathematical appendix at the end of this chapter.

THE BLENDING PROBLEM ONCE MORE

Since linear programming problems typically begin in the form of a set of requirements *in*equations, in this major section of the present chapter our first objective will be to get some insight into the defined nature of dual sets of inequations. Once one acquires such an insight, dual sets lose much of their mathematical mystery and the "remarkable" relationships we shall "discover" between dual sets of equations will be seen to be not so surprising after all.

But the matter will perhaps be more understandable if we do not try to solve our linear programming problem while leaving it in the form of inequations. Consequently, our second step in this section will be to turn the dual sets of inequations into dual sets of equations. At least superficially, this makes the problem appear more complicated and unwieldy than it really is. There is not much danger of our being appalled by this process, however, if we begin with an insight into the nature of dual sets of inequations, for we should then be able to see how the process of turning the dual sets of inequations into dual sets of equations does not really alter the fundamental nature of the problem, but simply puts the initial problem into a soluble form.

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[1] C. E. Lemke, "The Dual Method of Solving the Linear Programming Problem," *Naval Research Logistics Quarterly*, 1954, pp. 36–47.

The third step of this section will be to solve the dual sets of the equations derived in our second step. If we were not interested in acquiring some education about duality, it would be unnecessary, of course, for us to do more than simply to solve the maximization problem—for the minimization dual of our old friend, the refinery blending problem, is of no real importance to the blending problem. Furthermore, as we shall see, once one understands duality, if one does need the solutions to both a problem and its dual (as one frequently does), one can get both solutions simply by solving one of the dual sets. But this is by no means obvious, until one has first gone through the rather cumbersome process of solving both the problem and its dual, in order to trace the relationships between the dual sets of equations.

Dual sets of inequalities. The blending problem of Chapter I, stated as a typical linear programming problem (and as one in which pitch, visbreaker feed, and flux were denoted by y_1, y_2, and y_3, respectively), amounted to:

$$\begin{array}{rcl}
\textit{Specifically} & \qquad & \textit{More Generally} \\
2y_1 + y_2 - 2y_3 \leq 0 & & a_{11}y_1 + a_{12}y_2 + a_{13}y_3 \leq b_1 \\
y_1 + y_2 - 3y_3 \leq 0 & & a_{21}y_1 + a_{22}y_2 + a_{23}y_3 \leq b_2 \\
y_1 + y_2 \qquad\quad \leq 1{,}000 & & a_{31}y_1 + a_{32}y_2 + a_{33}y_3 \leq b_3 \\
\text{Maximize } 5y_1 + 4y_2 - 3y_3 = Z & & \text{Maximize } c_1y_1 + c_2y_2 + c_3y_3 = Z
\end{array}$$

This is a maximization problem in the form of requirements inequations, while its dual is a minimization problem in the same form (except that the inequation sign is reversed from $\leq$ to $\geq$).

The *dual* of the maximization problem looks like this:

$$\begin{array}{rcl}
\textit{Specifically} & \qquad & \textit{More Generally} \\
2x_1 + x_2 + \quad x_3 \geq 5 & & a_{11}x_1 + a_{21}x_2 + a_{31}x_3 \geq c_1 \\
x_1 + x_2 + \quad x_3 \geq 4 & & a_{12}x_1 + a_{22}x_2 + a_{32}x_3 \geq c_2 \\
-2x_1 - 3x_2 \qquad \geq -3 & & a_{13}x_1 + a_{23}x_2 + a_{33}x_3 \geq c_3 \\
\text{Minimize } 0x_1 + 0x_2 + 1{,}000x_3 = \not{Z} & & \text{Minimize } b_1x_1 + b_2x_2 + b_3x_3 = \not{Z}
\end{array}$$

A brief comparison of these dual sets of inequalities and their functionals, at either the specific or the more general level, reveals the defined nature of duality rather clearly. Indeed, given either statement of the two problems, the dual can be immediately formulated from the first statement.

Consider the three requirements inequations and the functional of the specific maximization problem. Ignoring the equality and inequality signs, we have a 4×4 matrix of coefficients (Z is also a coefficient, since there can be only one maximization value). If we stand at the bottom (or South) side of this matrix, we are looking at the problem from South's viewpoint, i.e., we need to assign a y_j value to each of the j columns of coefficients (except the far right column) in such a way as to satisfy the three inequations and maximize the functional.

The 4×4 matrix of coefficients for the dual of this maximization problem (namely, for the minimization problem) has exactly the same coefficients as has the maximization problem itself—with one *apparent* exception: the dual has the coefficient of $\not{Z}$, rather than Z, and this is actually not an exception, since it can be proved that $Z=\not{Z}$. The *position* of these coefficients is, however, different. In the dual, the first *column* of coefficients is the same as the first *row* of coefficients in the maximization problem; the second *column* in the dual is the same as the second *row* in the maximization problem, and so on. And the left side of the dual's inequations is $\geq$ the right side, whereas the left side of the inequations of the maximization problem is $\leq$ the right side. In brief, the dual matrix is, in matrix algebra terminology, simply the *transpose* of the original matrix.

This transposing of the original matrix amounts to our looking at the original matrix from a different viewpoint. In effect, it amounts to our switching the positions of the first and fourth rows and of the second and third rows, then turning the matrix clockwise by 90° and saying to ourselves, "Now we are looking at the matrix from East's viewpoint."

East's viewpoint being a different one geographically, and being also a minimization (rather than a maximization) one, we need to find x_i values for each of the i columns of coefficients (except the far right column) in such a way as to satisfy the three inequations and minimize the functional. In a very precisely defined sense, then, each of these two problems is the obverse (in the opponent sense) of the other, and the nature of this obverseness is what accounts for the seemingly remarkable comparisons we shall shortly discover between the two solutions.

It might be noted, however, that it is one thing to say that one might intuitively expect a relationship between these two problems and quite another to say that the nature of this relationship is obvious. To see the nature of this highly useful relationship, we must solve the maximization and minimization problems, and to do this we wish first to turn our dual sets of inequations into dual sets of equations.

Dual sets of equations. It will be recalled that, in Chapter I, we did turn the maximization problem's set of inequations into a set of equations, by adding slack (giveaway) variables, to get:

$$\begin{array}{rcl} 2y_1 + y_2 - 2y_3 + y_4 & & = 0 \\ y_1 + y_2 - 3y_3 \qquad + y_5 & & = 0 \\ y_1 + y_2 \qquad\qquad\qquad + y_6 & & = 1{,}000 \end{array}$$

By appropriate substitution, it can be shown that the functional to be maximized in this problem can also be stated:

$$\text{Maximize } - y_2 + 12y_3 - 5y_5$$

It will also be recalled from Chapter I that it was stipulated that the optimum solution ran in terms of y_2, y_3, and y_5 (i.e., in terms of visbreaker feed, flux, and the gravity giveaway factor) as the basic variables.

Now our major objective in this chapter is to see the relationship, in both specific and generalized form, between the dual sets of equations, when we have reached the *optimum* solution. It will be easier to see the nature of this relationship, if we renumber the variables of the maximization problem so that the basic variables of the optimum solution are changed from y_2, y_3, and y_5 to y_1, y_2, and y_3. When this is done, the above set of equations reads as follows:

Specifically

$$\begin{array}{rcl} y_1 - 2y_2 \qquad + 2y_4 + y_5 & & = 0 \\ y_1 - 3y_2 + y_3 + y_4 \qquad & & = 0 \\ y_1 \qquad\qquad + y_4 \qquad + y_6 & & = 1{,}000 \end{array}$$

$$\text{Maximize } - y_1 + 12y_2 - 5y_3$$

Generally

$$\sum_{j=1}^{n} a_{ij}y_j = b_i(i = 1, \cdots, m)$$

$$\text{Maximize } \sum_{j=1}^{n} c_jy_j$$

This being a maximization problem, we designate it as the Max-D (for Maximization Dual) to distinguish it from the Min-D (for Minimization Dual).

To get the Min-D of our problem, we need to do five things, some of which we also had to do to get the dual set of inequations:

(1) Transpose the a_{ij} coefficients of the Max-D, i.e., make the first *row* of the *Max-D* matrix of coefficients become the first *column* of the *Min-D* matrix, the second row of the Max-D matrix the second column of the Min-D matrix, and so on for additional rows in the Max-D. This results in making the $m \times n$ (3×6) matrix of the Max-D into an $n \times m$ (6×3) matrix for the Min-D.

(2) Assign a new variable, x_{n+i} $(i=1, \cdots, m)$, to each of these m *columns* of the Min-D matrix. In our case, this means assigning $x_{6+1}=x_7$, $x_{6+2}=x_8$, and $x_{6+3}=x_9$ to the coefficients of the three columns of the Min-D matrix.

We place no sign restriction on these variables, for reasons that will be clear under (4).

(3) Turn the n rows of this Min-D into inequalities by placing $c_j (j=1, \cdots, n)$ on the right side of an inequality sign for each of these rows, with the left side of each of these inequalities being $\geq$ its c_j.

Note that the right sides of these inequalities are the coefficients of the *functional* of the *Max-D problem.*

(4) *Subtract* one new variable (with a coefficient of unity), $x_j (j=1, \cdots, n)$, from each of the *n rows* of the Min-D, to turn the inequalities into equations: x_1 from the first row, x_2 from the second row, and so on. We specify that these x_j must all be *non-negative.* Because of the $\geq$ sign of requirement (3), and because of the specification that all x_j be non-negative, we cannot afford the luxury of a sign restriction on the x_{n+i}.

(5) Specify as a *minimization* functional for the Min-D problem that of minimizing

$$\sum_{i=1}^{m} b_i x_{n+i}.$$

In brief, the m coefficients of this minimization functional come from the right sides of the *m requirements equations* of the *Max-D problem.*

Performing these five operations on the Max-D, we get the following Min-D problem:

Specifically

$$\begin{array}{rrrrrrrrrr}
x_7 + & x_8 + x_9 - x_1 & & & & & & = & -1 \\
-2x_7 - 3x_8 & & - x_2 & & & & & = & 12 \\
& x_8 & & - x_3 & & & & = & -5 \\
2x_7 + & x_8 + x_9 & & & - x_4 & & & = & 0 \\
x_7 & & & & & - x_5 & & = & 0 \\
& x_9 & & & & & - x_6 & = & 0
\end{array}$$

Minimize $0x_7 + 0x_8 + 1{,}000x_9$

Generally

$$\sum_{i=1}^{m} a_{ij}x_{n+i} - x_j = c_j \quad (j = 1, \cdots, n)$$

Minimize $\sum_{i=1}^{m} b_i x_{n+i}$

In the Min-D, then, in the first three columns we have the same a_{ij} coefficients as in the three rows of the left side of the Max-D, but in the Max-D the i and j subscripts designate the rows and columns, respectively, in the familiar manner, rather than the reverse designation of the Min-D. This reverse designation is, of course, the result of transposing the Max-D matrix.

A terse and completely generalized statement of this dual set of problems is as follows:

Max-D	*Min-D*
Requirements Equations	
$\sum_{j=1}^{n} a_{ij}y_j = b_i(i = 1, \cdots, m)$	$\sum_{i=1}^{m} a_{ij}x_{n+i} - x_j = c_j(j = 1, \cdots, n)$
Restrictions	
All y_j non-negative	All x_j non-negative No sign restriction on x_{n+i}
Functional	
Maximize $\sum_{j=1}^{n} c_j y_j$	Minimize $\sum_{i=1}^{m} b_i x_{n+i}$

Solving the dual sets. The maximization problem poses no new difficulties for us. Its solution is:

$$\begin{matrix} c_1 = -1 \\ c_2 = 12 \\ c_3 = -5 \end{matrix} \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1/2 & 0 & 1/2 \\ -3/2 & 1 & 1/2 \end{bmatrix}$$

$$\begin{matrix} & & c_4 = 0 & & c_5 = 0 & & c_6 = 0 & \end{matrix}$$

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1{,}000 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} y_4 - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} y_5 - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} y_6 \right\}$$

$$= \begin{bmatrix} 1{,}000 \\ 500 \\ 500 \end{bmatrix} - \begin{bmatrix} 1 \\ -1/2 \\ -3/2 \end{bmatrix} y_4 - \begin{bmatrix} 0 \\ -1/2 \\ -3/2 \end{bmatrix} y_5 - \begin{bmatrix} 1 \\ 1/2 \\ 1/2 \end{bmatrix} y_6$$

$$z_{00} = 2{,}500 \quad z_{04} = -1/2 \quad z_{05} = -3/2 \quad z_{06} = -5/2$$

This is clearly an optimum solution:

$$\text{all } \overset{m}{\underset{u=1}{y_u}} > 0 \text{ and all } \overset{n-m}{\underset{k=1}{z_{0,m+k}}} \leq 0.$$

The minimization problem confronts us with a new type of situation. We have some variables which are not sign-restricted. We need to rid ourselves, at least on the surface, of these variables, so that we can state the problem purely in terms of the sign-restricted x_j variables. This can be done, as we shall now see.

The minimization problem deals with a system of n (six, in our case) equations in $n+m$ (nine, in our case) variables, and m (three, in our case) of these variables are not sign-restricted. By proper choosing of m of these n equations (i.e., three of the six equations), we can solve for the x_{n+i} (i.e., for x_7, x_8, and x_9) in terms of m of the x_j. These solutions for our three x_{n+i} can then be substituted in the remaining $(n-m)$ equations. This will leave us with three (or, $n-m$) equations in six (or, n) variables, namely, the x_j variables which are sign restricted.

Solving the first three equations of the Min-D for the x_{n+i}, we get:

$$\begin{bmatrix} x_7 \\ x_8 \\ x_9 \end{bmatrix} = \begin{bmatrix} g_{11} = 0 & g_{21} = -1/2 & g_{31} = -3/2 \\ g_{12} = 0 & g_{22} = 0 & g_{32} = 1 \\ g_{13} = 1 & g_{23} = 1/2 & g_{33} = 1/2 \end{bmatrix}$$

$$\cdot \left\{ \begin{bmatrix} -1 \\ 12 \\ -5 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x_3 \right\},$$

or,

$$\begin{aligned} x_7 &= 3/2 \qquad\qquad - 1/2x_2 - 3/2x_3 \\ x_8 &= -5 \qquad\qquad\qquad\quad + x_3 \\ x_9 &= 5/2 + x_1 + 1/2x_2 + 1/2x_3 \end{aligned}$$

Substituting the x_{n+i} solutions into the three remaining Min-D requirements equations and the functional, we get:

$$\begin{array}{rcl} -x_1 + 1/2x_2 + 3/2x_3 + x_4 & & = 1/2 \\ 1/2x_2 + 3/2x_3 \quad + x_5 & & = 3/2 \\ -x_1 - 1/2x_2 - 1/2x_3 \qquad\quad & + x_6 & = 5/2 \end{array}$$

$$\text{Minimize } 2{,}500 + 1{,}000x_1 + 500x_2 + 500x_3 = d_0 + d_1x_1 + d_2x_2 + d_3x_3$$

While this problem looks quite different from the one from which we have just derived it, since there is nothing mysterious or mathematically wrong about the substitution process used, this is still the same problem in somewhat different dress. Thus, for example, the initial functional of $0x_7+0x_8+1{,}000x_9$ does become $2{,}500+1{,}000x_1+500x_2+500x_3$, when we substitute for the x_9 of the initial functional the value we found for x_9 in terms of x_1, x_2, and x_3. One might also note, at this point, that the d_0, d_1, d_2, and d_3 are the z_{00}, y_1, y_2, and y_3 of our *max*imization problem.

A non-mathematician would scarcely suspect from the formulation of this problem that the minimizing solution is the one in which x_4 x_5, and x_6 are the basic variables, but such is the case. Since the inverse matrix of the matrix of the coefficients of these three variables is a unit matrix, we can immediately write down the solution as:

$$\begin{array}{c} d_4 = 0 \\ d_5 = 0 \\ d_6 = 0 \end{array} \quad \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} = \underset{\not{p}_{00} = z_{00} = 2{,}500}{\overset{d_0 = 2{,}500}{\begin{bmatrix} 1/2 = \not{p}_{40} = -z_{04} \\ 3/2 = \not{p}_{50} = -z_{05} \\ 5/2 = \not{p}_{60} = -z_{06} \end{bmatrix}}} - \underset{\not{p}_{01} = z_{10} = 1{,}000}{\overset{d_1 = 1{,}000}{\begin{bmatrix} -1 = \not{p}_{41} = -z_{14} \\ 0 = \not{p}_{51} = -z_{15} \\ -1 = \not{p}_{61} = -z_{16} \end{bmatrix}}} x_1$$

$$- \underset{\not{p}_{02} = z_{20} = 500}{\overset{d_2 = 500}{\begin{bmatrix} 1/2 = \not{p}_{42} = -z_{24} \\ 1/2 = \not{p}_{52} = -z_{25} \\ -1/2 = \not{p}_{62} = -z_{26} \end{bmatrix}}} x_2 - \underset{\not{p}_{03} = z_{30} = 500}{\overset{d_3 = 500}{\begin{bmatrix} 3/2 = \not{p}_{43} = -z_{34} \\ 3/2 = \not{p}_{53} = -z_{35} \\ -1/2 = \not{p}_{63} = -z_{36} \end{bmatrix}}} x_3$$

This is clearly an optimum (minimum) solution:

$$\text{all } \overset{n-m}{\underset{k=1}{x_{m+k}}} > 0 \text{ and all } \overset{m}{\underset{u=1}{\not{p}_{0u}}} \geq 0.$$

If we make a careful comparison of the various z terms in the Max-D and Min-D solutions, turning the *number* subscripts of those terms into their *general* form,[2] we shall see that four notable things are true:

(1) $\not{p}_{00} = z_{00}$; thus the optimum solution to the maximizing problem is the same as the optimum solution to the minimizing problem.

(2) $\not{p}_{m+k,0} = -z_{0,m+k}$; thus the $x_{m+k,0} = -z_{0,m+k}$.

(3) $\not{p}_{0u} = z_{u0}$; thus the $\not{p}_{0u} = y_u$.

(4) $\not{p}_{m+k,u} = -z_{u,m+k}$; thus the $\not{p}_{m+k,u}$ *columns* are identical with the $-z_{u,m+k}$ *rows.*

If none of these things is a remarkable coincidence resulting from our particular example, this means that one can write down the entire solution of either dual from the solution of the other. That these generalizations are not a coincidence is shown in the appendix, page 77.

THE FUNDAMENTAL DUALITY THEOREM

If the generalizations we have just drawn above are justifiable ones (and the appendix demonstrates that such is the case), we can immediately infer, from those generalizations, the fundamental duality theorem. As we shall learn in Chapters VII and VIII, this theorem is basic to game theory, and the fundamental duality theorem states that, given two dual sets: (1) if a finite solution to the maximization (minimization) functional exists, there is also a finite solution to the minimization (maximization) functional; and (2) the two optimum

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[2] The reason why renumbering of the variables of the maximization problem makes it easier to see the relationships between the optimum maximization and minimization solutions should now be easily understood. By this renumbering, it was possible, for example, to emphasize that the subscripts of the basic variables of the former problem's optimum solution were precisely the same as the subscripts of the *non*-basic variables of the latter problem's optimum solution. More generally, and phrased somewhat differently, renumbering the variables so that the basic variables in the final solution of our maximization problem are

$$\overset{m}{\underset{u=1}{y_u}}$$

means that the basic variables in the final solution of the dual are

$$\overset{n-m}{\underset{k=1}{x_{m+k}}}.$$

solutions will be equal. The appendix to this chapter is really a proof of this theorem; and, accepting for the moment the validity of that appendix as a justification for the above generalizations, we can readily see from those generalizations why this is true.

On reaching the optimum solution for the maximization problem, we necessarily had non-negative

$$\overset{m}{\underset{u=1}{y_u}} \text{ and negative } \overset{n-m}{\underset{k=1}{z_{0,m+k}}},$$

for these are the prerequisites for an optimum solution to a maximization problem. But, if our generalizations are valid, this is just another way of saying that

$$\overset{n-m}{\underset{k=1}{\not{p}_{m+k,0}}}$$

(which necessarily equal the $-z_{0,m+k}$) and the

$$\overset{m}{\underset{u=1}{\not{p}_{0u}}}$$

(which necessarily equal the z_{u0}) are non-negative. Since the $z_{0,m+k}$ are non-positive, this means non-negative values for both the $\not{p}_{m+k,0}$ (i.e., the basic variables, x, of the minimization problem) and the $\not{p}_{0u}$—and these are the prerequisites for an optimum solution to the minimization problem. Finally, since the $\not{p}_{00}$ and the z_{00} necessarily include the same terms, so that $z_{00}=\not{p}_{00}$, the optimum maximum must equal the optimum minimum.

From the viewpoint of the relationship between dual problems and zero-sum two-person games, it is helpful to realize that a more general statement than that of a zero-sum two-person game (to be considered in the next chapter) is actually implicit in equations (1a), (2a), (7), and (8).[3] To see why this is so, we need only consider these equations briefly.

Since each $z_{u0}=y_u\geq 0$ and each $-z_{0,m+k}=x_{m+k}\geq 0$, it is clear from equations (1a) and (7), respectively, that:

$$z_{u0} \geq \sum_{k=1}^{n-m} - z_{u,m+k}y_{m+k}, \quad \text{and,} \quad - z_{0,m+k} \geq \sum_{u=1}^{m} z_{u,m+k}x_u,$$

or,

---

[3] Equations (7) and (8) are the $f(x)$ counterparts of (1a) and (2a), and are to be found in the Appendix at the end of this chapter.

$$\sum_{k=1}^{n-m} - z_{u,m+k}y_{m+k} \leq z_{u0}, \quad \text{and} \quad \sum_{u=1}^{m} - z_{u,m+k}x_u \geq z_{0,m+k}$$

Similarly, it is clear from equations (2a) and (8) that we can specify:

$$\text{Maximize}\, f(y) - z_{00} = \sum_{k=1}^{n-m} z_{0,m+k}y_{m+k},$$

and

$$\text{Minimize}\, f(x) - z_{00} = \sum_{u=1}^{m} z_{u0}x_u$$

By a change of notation, these dual sets of inequalities and their functionals can be put in a form which makes it obvious that they represent a more generalized form than the form in which a zero-sum two-person game is stated in the next chapter.

Suppose we make the following change in notation:

$$\begin{array}{ll} n' = (n - m) & x_i = x_u \\ \alpha_{ij} = - z_{u,m+k} & \gamma_j = z_{0,m+k} \\ \beta_i = z_{u0} & f(x)' = f(x) - z_{00} \\ y_j = y_{m+k} & f(y)' = f(y) - z_{00} \end{array}$$

The dual sets of inequalities and their functionals can now be stated as:

$$\sum_{j=1}^{n'} \alpha_{ij}y_j \leq \beta_i \qquad \sum_{i=1}^{m} \alpha_{ij}x_i \geq \gamma_j$$

$$\text{Maximize}\, f(y)' = \sum_{j=1}^{n'} \gamma_j y_j \qquad \text{Minimize}\, f(x)' = \sum_{i=1}^{m} \beta_i x_i$$

This, as will be clear, is more general than a similar statement for zero-sum two-person games in the next chapter. For this reason, zero-sum two-person games are dual problems in the same sense as that in which a simplex tableau refers simultaneously to dual problems. This means, as will be shown in Chapter VIII, that the fundamental duality theorem provides us with a method of simultaneously solving the problem of both players in a zero-sum two-person game. Indeed, it also means that the fundamental duality theorem constitutes one (although by no means either the first or the only) proof of the basic theorem of the theory of games, namely, the minimax theorem.

APPENDIX TO CHAPTER VI

MAX-D AND MIN-D RELATIONSHIP GENERALIZED

The nature of the relationship between the z's of the Max-D and the $\not{z}$'s of the Min-D is the inevitable result of the five-step method by which we constructed the Min-D from the Max-D. The fact that many problems are at least potentially of this form (game theory problems, for example) indicates that these problems are more prevalent and more important than one might suppose, in view of the exacting requirements for the existence of a dual set of equations. It is desirable, therefore, to understand why our result was inevitable.

From Chapter III we know that, in the case of the Max-D, we have:

$$\overset{m}{\underset{u=1}{y_u}} = z_{u0} - \sum_{k=1}^{n-m} z_{u,m+k}y_{m+k} \tag{1a}$$

$$\text{Maximize } f(y) = z_{00} + \sum_{k=1}^{n-m} z_{0,m+k}y_{m+k} \tag{2a}$$

This is familiar enough to require no further elaboration, but the Min-D warrants some attention.

A logical starting point is to consider the process of eliminating the x_{n+i}, since we drew no broad generalizations from that point on in considering our problem. Referring to the solution for x_7, x_8, and x_9, it is apparent from the subscripts of the inverse matrix for x_7, x_8, and x_9, from the unit coefficient of x_1, x_2, and x_3, and from the subscripts of c_1, c_2, and c_3, that we can generalize the specific solution given for x_{n+i} on page 11 as:

$$\begin{bmatrix} x_{n+1} \\ \cdot \\ \cdot \\ \cdot \\ x_{n+m} \end{bmatrix} = \begin{bmatrix} g_{11} & \cdots & g_{m1} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ g_{1m} & \cdots & g_{mm} \end{bmatrix} \left\{ \begin{bmatrix} c_1 \\ \cdot \\ \cdot \\ \cdot \\ c_m \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} x_1 + \cdots + \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} x_m \right\} \tag{3}$$

or, more succinctly,

$$x_{n+i} = \sum_{u=1}^{m} (c_u + x_u)g_{uj}, \text{ for any } i = j = 1, \cdots, m \tag{3a}$$

Note that the g_{ij} inverse matrix of (3) is the *inverse* of the *transpose* of the 3×3 matrix of coefficients associated with the

$$\overset{m}{\underset{u=1}{y_u}}$$

of the maximization problem. The g_{uj} of (3a) represents any one of the j rows of this inverse matrix, but this j must equal the i of the subscript in x_{n+i}.

Similarly, the subscripts in the remaining $(n-m)$ equations (three, in our case) are such that we can generalize the substitution of the x_{n+i} into those equations as:

$$(4) \quad x_{m+k} = -c_{m+k} + \sum_{i=1}^{m} a_{i,m+k}x_{n+i}, \text{ for any } k = 1, \cdots, (n-m)$$

Here, $a_{i,m+k}$ is any one of the three $(m+k)$ columns of the 3×3 matrix of coefficients associated with the three y_{m+k} of our maximization problem. It is in equation (4) that our interest largely centers, in the sense that we want an alternative expression for its right side.

On the right side of (3), we have an alternative expression for the x_{n+i}, and there is also an obvious alternative expression for $-c_{m+k}$. Knowing from Chapter III that

$$z_{0,m+k} = c_{m+k} - \sum_{u=1}^{m} c_u z_{u,m+k},$$

we have:

$$(5) \qquad -c_{m+k} = -z_{0,m+k} - \sum_{u=1}^{m} c_u z_{u,m+k}$$

Derivation of an alternative expression for

$$\sum_{i=1}^{m} a_{i,m+k}$$

is less obvious.

Let us begin by considering, from our specific problem, the sum of $(z_{14}+z_{24}+z_{34})$ as representative of the generalized sum of

$$\sum_{u=1}^{m} z_{u,m+k}.$$

The properties of the elements of the inverse matrix entering into the determination of the $z_{u,m+k}$ are such that:

$$(a_{14} + a_{24} + a_{34}) = (a_{11}z_{14} + a_{12}z_{24} + a_{13}z_{34}) + (a_{21}z_{14} + a_{22}z_{24} + a_{23}z_{34}) \\ + (a_{31}z_{14} + a_{32}z_{24} + a_{33}z_{34})$$

If we substitute the specific values of our problem, for example, for these letter coefficients, we get:

$$(2 + 1 + 1) = (1 + 1 + 0) + (1 + 3/2 - 3/2) + (1 + 0 + 0)$$

Stated generally, this means that:

$$\text{(6)} \qquad \sum_{i=1}^{m} a_{i,m+k} = [z_{1,m+k} \cdots z_{m,m+k}] \times \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ \cdot & & \\ \cdot & & \\ a_{1m} & \cdots & a_{mm} \end{bmatrix}$$

Note that the a_{ij} matrix on the right is the *transpose* of the 3×3 matrix of coefficients associated with the

$$\overset{m}{\underset{u=1}{\nu_u}}$$

of the maximization problem.

Since substitution of (3), (5), and (6) into (4) involves the product of the right sides of (3) and (6), and since the $[g_{ij}]$ matrix of (3) is simply the inverse of the $[a_{ij}]$ matrix of (6), the substitution results in:

$$x_{m+k} = \left(-\, z_{0,m+k} - \sum_{u=1}^{m} c_u z_{u,m+k}\right) + \sum_{u=1}^{m} z_{u,m+k}(c_u + x_u), \quad \text{or}$$

$$\text{(7)} \quad x_{m+k} = -\, z_{0,m+k} + \sum_{u=1}^{m} z_{u,m+k} x_u, \text{ for any } k = 1, \cdots, (n - m)$$

Likewise, if we substitute the right side of (3) into the functional $(b_1y_{n+1} + \cdots + b_my_{n+m})$ to be minimized, we have:

$$\text{Minimize } f(x) = \begin{bmatrix} b_1g_{11} & \cdots & b_1g_{m1} \\ \cdot & & \\ \cdot & & \\ b_mg_{1m} & \cdots & b_mg_{mm} \end{bmatrix}$$

$$\left\{ \begin{bmatrix} c_1 \\ \cdot \\ \cdot \\ \cdot \\ c_m \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} x_1 + \cdots + \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} x_m \right\}$$

As can be readily verified, however, the product of the $[b_i g_{ij}]$ and $[c_j]$ matrices necessarily includes exactly the same terms as does the z_{00} of the Max-D. Moreover, the product of the $[b_i g_{ij}]$ and each of the remaining $(m \times 1)$ matrices is obviously $(b_1 g_{11} + \cdots + b_m g_{1m}), \cdots,$ $(b_1 g_{m1} + \cdots + b_m g_{mm})$—and these are precisely the terms also for $z_{10}, \cdots, z_{m0}$. Hence, the minimization functional can be expressed as:

$$\text{(8)} \qquad \text{Minimize } f(x) = z_{00} + \sum_{u=1}^{m} z_{u0} x_u$$

Equations (7) and (8) are, of course, the Min-D counterparts of equations (1a) and (2a) of the Max-D. Thus we have been able to derive (7) and (8) in such a way that all symbols in them, except the x_j, correspond to symbols found in (1a) and (2a). From a comparison of (7) and (8) with (1a) and (2a), therefore, it is clear that the relationships we found between the z's and the p's of our specific problem were more than a remarkable coincidence—they were inevitable.

CHAPTER VII

Game Theory

Zero-sum two-person games are, in at least one sense, a special case of linear programming problems. But it is equally valid to say that, in another sense, such games are more complicated than linear programming problems. Solving a linear programming problem involves solving a set of equations for non-negative variables in such a way as to maximize (minimize) some functional. In doing so, a solution to the dual set of equations is automatically obtained; but this solution to the dual set of equations is incidental, rather than being a fundamental objective.

Solving a game also amounts to solving a set of equations for non-negative variables in such a way as to maximize (minimize) some functional. Unlike a linear programming problem, however, obtaining solutions to both of the dual sets of equations is the fundamental objective in solving a game. In brief, the solution to both of the dual sets is indispensable to *the* solution of a game; while, in linear programming, the solution to one of the dual sets is generally only an interesting by-product of getting the solution to that particular dual set in which one is interested.

This comparison largely follows from the underlying philosophy of game theory as to what constitutes the best strategy on the part of the opponents in a game. For this philosophy underlies the Minimax Theorem, and the Minimax Theorem accounts for the importance of duality and linear programming as a method of solving games. Consequently our focal points of interest in this chapter are the major concepts and philosophy of game theory, and, given that philosophy, the validity of the Minimax Theorem. The problem of solving a game by turning it into a linear programming problem (and, conversely, of solving a linear programming problem by turning it into a game) then follows logically in Chapters VIII and IX.

MAJOR CONCEPTS

Two basic premises of zero-sum two-person games are that (1) both players are malevolent (i.e., each is concerned with maximizing his own gains, or minimizing his own losses) and (2) both are aware of, and intelligent enough to evaluate accurately, the payoffs associated with both players' alternative strategies. Assump

The payoffs associated with the players' alternative strategies can be expressed in the form of a *payoff matrix.* Thus the 2×2 matrix below means that B can choose either ${}_sB_1$ or ${}_sB_2$, the two pure strategies associated with the two payoff columns, while A can choose either ${}_sA_1$ or ${}_sA_2$, the two pure strategies associated with the two payoff rows. The payoff will be the number at the point of juncture of their two choices. If B chooses ${}_sB_2$ and A chooses ${}_sA_1$, the payoff will be $a_{12}=2$.

If the numbers represent the amount A receives, A would then win 2, while B would lose an equal amount. There being no negative numbers in the matrix, this is a singularly unprofitable game for B; but, in the interests of simplicity and with a philanthropic spirit towards education, B consents to play for us—reserving the right, however, to do his best to minimize his losses.

Matrix I

(Payoffs to A)

		B		
		${}_sB_1$	${}_sB_2$	Row Min.
A	${}_sA_1$	$a_{11} =$ 3	$a_{12} =$ 2	2*
	${}_sA_2$	$a_{21} =$ 2	$a_{22} =$ 1	1
Col. Max.		3	2*	

Now if A's strategies are associated with the payoff rows and A wants to maximize his gains, since A knows B wishes to minimize his own losses (which are A's gains), A will be interested in inspecting the *minimum* in each of his rows. These row minima appear to the right of the matrix. Moreover, since A wants to maximize his gains, he will choose that strategy which is associated with the largest minimum—namely, ${}_sA_1$. In brief, A strives for the maximum of the minima—or, the *maximin.*

Conversely, if B's strategies are associated with the payoff columns and B wants to minimize his losses, since B knows A wishes to maximize his own gains (which are B's losses), B will inspect the maximum in each of his columns. These column maxima are shown at the bottom of the matrix. Since B wants to minimize his losses, he will choose that strategy which is associated with the smallest maximum —namely, ${}_sB_2$. Consequently, B always strives for the minimum of the maxima—or, the *minimax.*

These choices by A and B do give us the juncture of $a_{12}=2$ as the payoff, in the above game. Furthermore, this payoff is also the *value of the game,* since A and B will always choose ${}_sA_1$ and ${}_sB_2$, respectively, in this particular game. Hence, in this particular game, the a_{12} payoff also happens to represent the value of the game, or the expected value of A's gains and B's losses.

The *solution* to a game is a statement of the odds (or probabilities) which each player should assign to each of his strategies. Thus the solution to the above game is: for A, 1.0:0.0; and for B, 0.0:1.0, meaning that A will maximize his gains by assigning a 1.0 probability coefficient to ${}_sA_1$ and that B will minimize his losses by assigning a probability coefficient of 1.0 to ${}_sB_2$.

Now this particular game is a rather unique one in two respects. In the first place, it has a *saddle point* of $a_{12}=2$. That is to say, the maximum of the row minima and the minimum of the column maxima are identically equal to 2. Secondly, *dominance* is present for both A and B. In A's case, ${}_sA_1$ dominates ${}_sA_2$—i.e., ${}_sA_1$ is superior to ${}_sA_2$ in *both columns,* and A would patently be foolish ever to play ${}_sA_2$. And in B's case, ${}_sB_2$ dominates ${}_sB_1$, so B would be foolish ever to play ${}_sB_1$. This does not mean, parenthetically, that dominance is confined to games with a saddle point; for dominance can also exist in a game which has no saddle point.

Given a game with a saddle point, finding the solution is not difficult, and that solution will amount simply to finding that pure (i.e., single) strategy for each player which will result in the maximin equalling the minimax. Given a game with no saddle point, however, finding the solution is all too frequently something less than easy. As we shall see, however, in such games a *mixed strategy* (i.e., a combination of pure strategies with a given frequency)[1] will result in the same stability (i.e., A's maximin $=$ B's minimax) as in games with a saddle point.

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[1] Even a pure strategy can, of course, be viewed as a mixed strategy with zero probability coefficients assigned to all but the pure strategy played.

MATHEMATICAL PHILOSOPHY

If neither dominance nor a saddle point is present, even so simple a game as that provided by a 2×2 payoff matrix serves to bring out the more important philosophical features of game theory. Our second matrix represents such a game.

This matrix is accompanied, in addition to the indication of the row minima and the column maxima, by a statement of the solution to the game. Our primary interest lies in this solution. More precisely, we are interested in assuring ourselves that the solution to a game, in the general case, necessarily means equality between the minimax and the maximin.[2]

Matrix II

(Payoffs to *A*)

$$\begin{array}{ll|c|c|cc}
 & & \multicolumn{2}{c}{B} & \text{Row} & A\text{'s} \\
 & & {}_sB_1 & {}_sB_2 & \text{Min.} & \text{Probabilities} \\
\cline{3-4}
A & {}_sA_1 & a_{11} = 1 & a_{12} = 5 & 1 & 0.2 = x_1 \\
\cline{3-4}
 & {}_sA_2 & a_{21} = 3 & a_{22} = 2 & 2^* & 0.8 = x_2 \\
\cline{3-4}
\multicolumn{2}{l}{\text{Col. Max.}} & 3^* & 5 & & \\
\multicolumn{2}{l}{B\text{'s Probabilities:}} & 0.6 = y_1 & 0.4 = y_2 & &
\end{array}$$

First, however, we note that there is no saddle point. This is clear from the fact that the pure maximin ($a_{22} = 2$) is smaller than the pure minimax ($a_{21} = 3$). It seems clear that mixed strategies are preferable to pure strategies, in this game. Suppose, for example, that *B* always elected ${}_sB_1$, his pure strategy associated with the pure minimax column. A would inevitably become aware of this constant pattern in *B*'s play, and *A* would always adopt ${}_sA_2$, his pure strategy associated with the maximin row, thus assuring himself of a gain of 3. But then B would inevitably become aware of this constant pattern in A's play, and B would shift to ${}_sB_2$ (even though this pure strategy is not associated with the pure minimax column), thus assuring himself of holding his losses down to 2. This, in turn, would lead to a shift in A's strategy, and so on. One's intuitive reaction to all this is that mixed strategies are preferable to pure strategies and that the value of the game must lie somewhere between the two values represented by the pure maximin and the pure minimax; and this is a valid conjecture, since the value of this game is 13/5.

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[2] This does not preclude the possibility of a game having more than one solution which will do this, although all solutions lead to the same value for the game, since there can be only one minimax and only one maximin.

At this point, however, it is certainly not obvious that the value of this game is precisely 13/5. For this can be obvious only after we have rigorous definitions for minimax and maximin and demonstrate that they must be equal. It is, however, easily seen that, *given the mixed strategies for A and B indicated above by A's and B's probabilities*, the average *payoff* is 13/5—even though we do not yet know that this is a stable value. To see this clearly, let us consider it both generally and specifically.

We can represent any mixed strategy of A symbolically as $x=(x_1, \cdots, x_m)$, where all

$$\overset{m}{\underset{i=1}{x_i}}$$

are non-negative and where $(x_1+ \cdots +x_m)=1$; and we can represent any mixed strategy of B as $y=(y_1, \cdots, y_n)$, where all

$$\overset{n}{\underset{j=1}{y_j}}$$

are non-negative and where $(y_1+ \cdots +y_n)=1$. The

$$\overset{m}{\underset{i=1}{x_i}} \text{ and the } \overset{n}{\underset{j=1}{y_j}}$$

are thus probability coefficients representing the relative frequency with which A and B elect to play each of their pure strategies. Consequently, the average payoff is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j a_{ij}.$$

In the above matrix, this amounts to $1/5 \cdot 3/5 \cdot 1 + 1/5 \cdot 2/5 \cdot 5 + 4/5 \cdot 3/5 \cdot 3 + 4/5 \cdot 2/5 \cdot 2 = 13/5$.

This does not necessarily mean, however, that the

$$\overset{m}{\underset{i=1}{x_i}}$$

and the

$$\overset{n}{\underset{j=1}{y_j}}$$

chosen by A and B are necessarily the strategies which will maximize A's gains and minimize B's losses. A's gains obviously have to equal B's losses. But just any

$$\overset{n}{\underset{i=1}{y_j}}$$

will not be the mixed strategy which minimizes B's losses, in which case B may be letting A get away with more than he should; and just any

$$\overset{m}{\underset{i=1}{x_i}}$$

will not be the mixed strategy which maximizes A's gains, in which case A may be letting B get off more cheaply than he should. The basic problem, then, is one of determining what is the most rational way for A and B to choose their mixed strategies.

Consider the problem first from A's viewpoint. In choosing a *particular* mixed strategy $\hat{x}=(\hat{x}_1, \cdots, \hat{x}_m)$, A credits B with the intelligence to choose that *particular* mixed strategy $\bar{y}=(\bar{y}_1, \cdots, \bar{y}_n)$ which will minimize the payoff,

$$\sum_i \sum_j \hat{x}_i \bar{y}_j a_{ij},$$

once he realizes that A's strategy is $\hat{x}$. This payoff, to illustrate with our 2×2 matrix, can also be expressed as:

$$\bar{y}_1(\hat{x}_1 a_{11} + \hat{x}_2 a_{21}) + \bar{y}_2(\hat{x}_1 a_{12} + \hat{x}_2 a_{22}) \tag{1}$$

But the two expressions in parentheses are the two column payoffs associated with B's two pure strategies, with each element of the matrix multiplied by the appropriate component of $\hat{x}$. Furthermore, A may choose the components of $\hat{x}$ in such a general way that (a) the first parenthetical expression is $>$ the second parenthetical expression

($\hat{x}_1 = 1/10$ and $\hat{x}_2 = 9/10$, for example, would result in the first expression being 28/10 and the second expression being 23/10), or, (b) the first parenthetical expression is < the second parenthetical expression ($\hat{x}_1 = \hat{x}_2 = 1/2$, for example, would result in the first expression being 4/2 and the second expression being 7/2), or, (c) the two parenthetical expressions are equal ($\hat{x}_1 = 1/5$ and $\hat{x}_2 = 4/5$, for example, results in both expressions being 13/5).

Since $\bar{y}_1 + \bar{y}_2 = 1$, the payoff

$$\left(\sum_i \sum_j \hat{x}_i \bar{y}_j a_{ij}\right),$$

is just a weighted average of amounts

$$\sum_i \hat{x}_i a_{ij}$$

multiplied by $\bar{y}_j$ weights summing to unity. Therefore, the payoff certainly cannot be smaller than the smaller of the two above parenthetical expressions. This, however, is just another way of saying that there is at least one pure strategy of B that will minimize his losses as well as any mixed strategy will, *given A's mixed strategy.* More generally, there will be at least one j for which

$$\sum_i \hat{x}_i a_{ij} \leq \sum_i \sum_j \hat{x}_i \bar{y}_j a_{ij}$$

or (in words) at least one pure strategy of B that will minimize his losses at least as well as will any mixed strategy B can choose, *given A's mixed strategy.*

This means that all A need do, in considering each of his own mixed-strategy possibilities, is to note that particular *pure* strategy of B which gives B the smallest loss in connection with A's particular *mixed* strategy. Since

$$\sum_i \hat{x}_i a_{ij} \ (j = 1, \cdots, n),$$

represent the n column payoffs associated with B's n pure strategies multiplied by the appropriate components of A's particular mixed strategy of $\hat{x}$, we can denote the minimum loss strategy of B as:

$$\min_j \sum_i \hat{x}_i a_{ij} = \sum_i \hat{x}_i a_{ij(\hat{x})}, \tag{2}$$

the $j(\hat{x})$ representing that pure strategy of B yielding the minimum loss, *when $\hat{x}$ is given.* To illustrate with reference to the three examples immediately following the payoff definition (1), these minimum-loss pure strategies for B would be:

(a) where $\hat{x}=(\hat{x}_1=1/10; \hat{x}_2=9/10$:
B's pure strategy ${}_sB_2$, because:

$$\min_{2} \sum_{i=1}^{2} \hat{x}_i a_{i2} = \sum_i \hat{x}_i a_{i2} = 23/10 < \sum_i \hat{x}_i a_{i1} = 28/10$$

(b) where $\hat{x}=(\hat{x}_1=\hat{x}_2=1/2)$:
B's pure strategy ${}_sB_1$, because:

$$\min_{1} \sum_{i=1}^{2} \hat{x}_i a_{i1} = \sum_i x_i a_{i1} = 4/2 < \sum_i \hat{x}_i a_{i2} = 7/2$$

(c) where $\hat{x}=(\hat{x}_1=1/5; \hat{x}_2=4/5)$:
Either of B's two pure strategies, since both yield identical losses of 13/5.

Now A clearly wants to maximize this minimum pure strategy loss,

$$\min_j \sum_i \hat{x}_i a_{ij},$$

of B, so A will choose his mixed strategy

$$\overset{m}{\underset{i=1}{x_i}}$$

to do this [which means the

$$\overset{m=2}{\underset{i=1}{\hat{x}_i}}$$

associated with (c) above]. Thus, we can define the maximin in which A is interested as η_1:

$$\max_x \min_j \sum_i x_i a_{ij} = \max_x \sum_i x_i a_{ij(x)} = \eta_1 \tag{3}$$

This mixed-strategy maximin can clearly not be smaller than the pure maximin (i.e., $\max_i \min_j a_{ij}$), which, in our present example, is 2 (see a_{22} of Matrix II). In other words, in our example, we know that $\eta_1 \geq 2$.

Having considered the problem in detail from A's viewpoint, we need hardly do more than write down the results we should get from considering the problem from B's viewpoint. B, in choosing his particular strategy $\hat{y}=(\hat{y}_1, \cdots, \hat{y}_n)$, will realize that A has at least one pure strategy

$$\sum_j \hat{y}_j a_{ij} \geq \sum_i \sum_j \bar{x}_i \hat{y}_j a_{ij}.$$

Since

$$\sum_j \hat{y}_j a_{ij}, \ (i = 1, \cdots, m),$$

represent the m row payoffs associated with A's m pure strategies, multiplied by the appropriate components of B's particular mixed strategy of $\hat{y}$, the maximum gain strategy of A is:

$$(4) \qquad \max_i \sum_j \hat{y}_j a_{ij} = \sum_j \hat{y}_j a_{i(\hat{y})j},$$

the $i(\hat{y})$ standing for the pure strategy of A yielding the maximum gain, when $\hat{y}$ is given.

B wants to minimize this maximum pure strategy gain,

$$\max_i \sum_j \hat{y}_j a_{ij},$$

of A, and B chooses his mixed strategy

$$\overset{n}{\underset{j=1}{y_j}}$$

to do this. Hence, the minimax (η_2) in which B is interested becomes:

$$(5) \qquad \min_y \max_i \sum_j y_j a_{ij} = \min_y \sum_j y_j a_{i(y)j} = \eta_2$$

This mixed-strategy minimax can clearly not be greater than the pure minimax (i.e., $\min_j \max_i a_{ij}$), which, in our present example, is 3 (see a_{21} of Matrix II). In other words, in our example, we know that $\eta_2 \leq 3$.

It is now clear that our intuitive feeling (i.e., that the value of the game must fall somewhere between 2 and 3) was justified. For we have the facts that $\eta_1 \geq 2$ and that $\eta_2 \leq 3$. None of this assures us, however, that the value of the game must be precisely 13/5 and that this is a stable value. These things can be true only if it can be shown that always $\eta_1 = \eta_2$.

THE MINIMAX THEOREM

The minimax theorem states that always $\eta_1 = \eta_2$, i.e., that always the maximin = the minimax. Equipped with the proper definitions and philosophy, we are now in a position to follow a proof of the minimax theorem.[3] This proof involves showing that:

(1) $\eta_1 \leq \eta_2$, which is easily done; and

(2) $\eta_1 \geq \eta_2$, which is less easy, since it requires invoking the theorem of the supporting hyperplanes and the theorem of the alternative for matrices.

From (1) and (2), it then follows that $\eta_1 = \eta_2$.

$\underline{\eta_1 \leq \eta_2.}$ We have seen, from the immediately preceding pages, that:[4]

(a) $$\min_j \sum_i \hat{x}_i a_{ij} \leq \sum_i \sum_j \hat{x}_i \bar{y}_j a_{ij}$$

(Given $\hat{x}$, B has at least one pure strategy which will minimize his losses as well as, or better than, any mixed strategy, $\bar{y}$, he can choose.)

(b) $$\max_i \sum_j \hat{y}_j a_{ij} \geq \sum_i \sum_j \bar{x}_i \hat{y}_j a_{ij}$$

(Given $\hat{y}$, A has at least one pure strategy which will maximize his gains as well as, or better than, any mixed strategy, $\bar{x}$, he can choose.)

As we know, A wants to maximize the left side of (a), and we defined this maximin in (3) on page 88 as

$$\max_x \min_j \sum_i x_i a_{ij} = \max_x \sum_i x_i a_{ij(x)} = \eta_1$$

But the only way that A can be sure of doing this is to choose that $\hat{x}$ which makes it a matter of indifference to B which pure strategy B chooses. Indeed, this is precisely why A should choose alternative (c) on page 88. It follows, then, that we can impute to A that particular strategy $\hat{x}$ represented by the left side of (a), which will, in turn, equal η_1. And, by similar reasoning, with reference to the minimax definition in (5) on page 89, we can assume B's choice of $\hat{y}$ will be the one represented by the left side of (b), so that η_2 = the left side of (b).

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[3] The first proof (and a rather difficult one) was given in 1928 by the late John von Neumann, "Zur Theorie der Gesellschaftsspiele," *Math. Annalen*, 1928. A more elementary proof, that given in this chapter, is to be found in von Neumann and Morgenstern, *Theory of Games and Economic Behavior* (Princeton, New Jersey: Princeton University Press, 1944), pp. 134–143; pp. 153–155. Other proofs also exist.

[4] A symbolic statement is just so much gobbledegook to anyone, until he thoroughly absorbs the meaning the statement is intended to convey. For the convenience of the reader, therefore, a parenthetical reminder of the meaning of (a) and (b) is included immediately below (a) and (b).

But if η_1 and η_2 = the left sides of (a) and (b), respectively, and if *A* and *B* choose the particular strategies just indicated of $\hat{x}$ and $\hat{y}$, we have that:

$$\text{(a}')\qquad \eta_1 \leq \sum_i \sum_j \hat{x}_i \hat{y}_j a_{ij}$$

$$\text{(b}')\qquad \eta_2 \geq \sum_i \sum_j \hat{x}_i \hat{y}_j a_{ij}$$

Since the right sides of inequalities (a′) and (b′) are identical, it follows that η_1 cannot be greater than η_2, so we have proved that $\eta_1 \leq \eta_2$.

$\underline{\eta_1 \geq \eta_2.}$ To show that $\eta_1 \geq \eta_2$ is a good deal more difficult than to show that $\eta_1 \leq \eta_2$. The difficulties may seem less complicated, however, if we begin with some general knowledge of how each successive step involved contributes to the conclusion that $\eta_1 \geq \eta_2$. We shall consider (A) the theorem of the supporting hyperplanes, because this is important to a part of the proof of (B) the theorem of the alternative for matrices. From the theorem of the alternative for matrices, it is easily shown (C) that it is impossible simultaneously that $\eta_1 < 0$ and $\eta_2 > 0$, or, phrased differently, that it is impossible that $\eta_1 < 0 < \eta_2$. It is then easy to show (D) that it is equally impossible for $\eta_1 < k < \eta_2$, for any value of k, so that η_1 can never be smaller than η_2; hence $\eta_1 \geq \eta_2$. Since we have previously seen that $\eta_1 \leq \eta_2$, it must follow that $\eta_1 = \eta_2$.

(A) THEOREM OF THE SUPPORTING HYPERPLANES

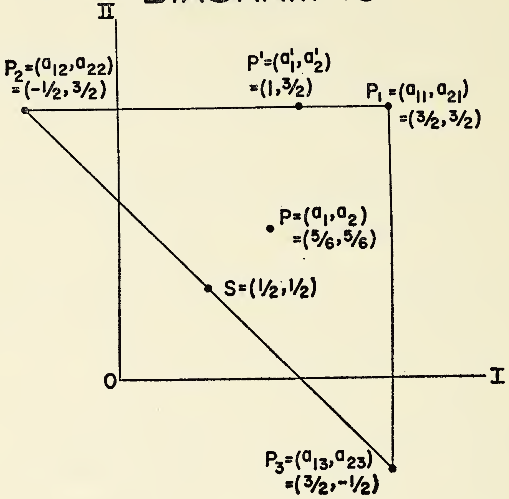
Even though we approach the supporting-hyperplanes and the alternative-for-matrices theorems in a relatively informal manner, calling rather heavily upon our intuition, we shall still be forced to operate on a somewhat formidable level of abstraction, if we are to acquire an understanding of the proof of the minimax theorem. Since such an understanding requires only that we be able to follow that proof (rather than that we be able to reproduce it from memory), however, this is not an insurmountable obstacle.

Before turning to the supporting-hyperplanes theorem proper, it will be helpful if we lay three bits of foundation. The first of these will be of obvious familiarity. The second and third of these, if they are not immediately of obvious familiarity, will soon be recognized as such.

Our first bit of foundation is that we begin with given values for a_{ij} $(i=1, \cdots, m;\ j=1, \cdots, n)$—thus, in Diagram 10, we have

$m=2$ and $n=3$. Then a *set* $(a_{1j}, \cdots, a_{mj})$ is the point P_j in m-dimensional space; thus, for example, the set $(a_{13}, a_{23}) = 3/2, -1/2)$ is the point P_3 in two-dimensional space. Diagram 10 is clearly an old friend, in many respects, since it is basically the same as Diagram 3 on page 18 of Chapter II.

DIAGRAM 10



The further point $P=(a_1, \cdots, a_m)$ belongs to the *convex hull* of $P_1, \cdots, P_n$, if we can find non-negative values for $t_1, \cdots, t_n$, where $t_1 + \cdots + t_n = 1$, such that

$a_i = t_1 a_{i1} + \cdots + t_n a_{in}$, for all $i=1, \cdots, m$ (i.e., for each *row* of a_{ij} coefficients)

The point P, and the condition $a_i = t_1 a_{i1} + \cdots + t_n a_{in}$, are more familiar than they may appear to be.

It will help to see this, if we get a good intuitive notion of what is meant more generally by a convex hull. In two-dimensional space, if two points are in a convex hull, so are all points on the straight line between these two points. Thus, for example, all points on a straight line drawn between S and P_1 of Diagram 10 would obviously be in the convex hull formed by $P_1P_2P_3$. And in three-dimensional space, if three points are in the convex hull, so are all other points in the triangular plane delineated by these three points. Thus, for example, in Diagram 4 on page 20 of Chapter II, the points $\alpha_m P_0$, $\alpha_M P_0$, and P_2 all lie in the tetrahedron $P_1P_2P_3P_4$ and these three points form a triangular plane within the tetrahedron, so all points of this triangular plane also lie in the convex hull of $P_1P_2P_3P_4$.

In terms of Diagram 10, to return to a two-dimensional convex hull, the point $P=(a_1, a_2)=(5/6, 5/6)$ belongs (as is visually obvious) to the convex hull P_1, P_2, P_3, and we can find non-negative values for $t_1, \cdots, t_3$, where $t_1+\cdots+t_3=1$, such that

$$a_1 = t_1a_{11} + t_2a_{12} + t_3a_{13}$$
$$a_2 = t_1a_{21} + t_2a_{22} + t_3a_{23}$$

or,

$$5/6 = 1/3\cdot 3/2 + 1/3\cdot -1/2 + 1/3\cdot 3/2$$
$$5/6 = 1/3\cdot 3/2 + 1/3\cdot 3/2 + 1/3\cdot -1/2$$

for $t_1=t_2=t_3=1/3$.

At this point, let us get some intuitive feeling for what this $t_1+\cdots+t_n=1$ actually constitutes. In effect, these t_j are merely weights (adding to unity) to be applied (1) to each of the horizontal components (a_{11}, a_{12}, and a_{13} in Diagram 10) of P_1, P_2, and P_3 to get the horizontal component of P, and (2) to each of the vertical components (a_{21}, a_{22}, and a_{23}) of P_1, P_2, and P_3 to get the vertical component of P.

Furthermore, the manner in which these weights are determined is easily understood, if we recall that, by the definition of a convex hull, all points on a straight line between two points in that hull are also in that same hull. Imagine, for example, a straight line drawn from P_1 to S, in Diagram 10. This diagram is so constructed that P also lies on this imaginary line of P_1S. Now the line segment SP has a length of $\sqrt{2}/3$, while the length of SP_1 is $\sqrt{2}$. In brief, $SP=SP_1/3$. In a mechanical-analogy sense, then, P_1 is "pulling" P one-third of the distance from S to P_1. This is why $t_1=1/3$. Similarly, it so happens in this particular case that both P_2 and P_3 are also exerting "pulls" upon P of one-third each. As we know from Chapter II, it is no problem to generalize this straight-line distance concept between points to include three (and higher) dimensions.

Now consider the point $P'=(a_1', a_2')=1, 3/2)$, which also belongs (by the skin of its teeth) to the convex hull $P_1P_2P_3$, since

$$a_1' = 3/4\cdot 3/2 + 1/4\cdot -1/2 + 0\cdot 3/2 = 1$$
$$a_2' = 3/4\cdot 3/2 + 1/4\cdot 3/2 + 0\cdot -1/2 = 3/2$$

for $t_1=3/4$; $t_2=1/4$; $t_3=0$.

In the mechanical-analogy sense, here P_3 obviously has no "pull" on P', since P' lies on the line P_1P_2. Consequently, $t_3=0$. But $t_1=3/4$ and $t_2=1/4$, because P_1 has "pulled" P' three-fourths of the distance away from P_2, while P_2 has "pulled" P' one-fourth of the distance away from P_1.

Now we can find a real similarity between the P and the $(t_1, t_2, t_3) = (1/3, 1/3, 1/3)$ of Diagram 10, on the one hand, and the P_0 and the $(y_1, y_2, y_3)=(7/10, 7/10, 1/2)$ entering into the

$$\alpha P_0 = \frac{10}{19} P_0$$

of Diagram 3 in Chapter II, on the other hand. The y_1, y_2, and y_3 of Diagram 3 actually constitute one set (of an infinite number of sets) of weights, *not* adding to unity, which takes P_1, P_2, and P_3 into P_0. The reason y_1, y_2, and y_3 add to more than unity is because P_0 lies outside of $P_1P_2P_3$. If we "shrink" the sum of y_1, y_2, and y_3 to unity, we "pull" P_0 into $P_1P_2P_3$, in the sense that no P_0 lying outside of $P_1P_2P_3$ will satisfy the two requirements equations on page 17 of Chapter II, when the y_1, y_2, and y_3 "weights" sum to unity. This is highly germane to our present objective, since we shall ultimately wish to think of our t_j weights as y_j.

With this foundation, we are prepared to turn to the theorem of the supporting hyperplanes. As we do so, let us keep in mind that our ultimate objective is to prove that $\eta_1 \geq \eta_2$, and that we are developing the proofs of the theorems of the supporting hyperplanes and of the alternative for matrices first to show that it is impossible simultaneously for $\eta_1 < 0$ and $\eta_2 > 0$.

The theorem of the supporting hyperplanes states that if the point $0=(0, \cdots, 0)$ is *not* in the convex hull $P_1, \cdots, P_n$, then there exist values $s_1, \cdots, s_m$ such that, for any point P in the convex hull, we have

$$s_1a_1 + \cdots + s_ma_m > 0$$

where $S=(s_1, \cdots, s_m)=$ the point in $P_1, \cdots, P_n$ for which the square of the "distance" from 0 (that is, the square root of the sum of squares of P's coordinates) is smallest.

This is fairly obvious (at least intuitively) for $m=2$. Thus, in Diagram 10, $S=(1/2, 1/2)$ is the closest point in $P_1P_2P_3$ to 0. And $s_1a_1+s_2a_2>0$ for any (a_1, a_2) coordinates in $P_1P_2P_3$. Furthermore, if the reader tries juggling the $P_1P_2P_3$ triangular plane around, while keeping the point (0, 0) out of the hull, he should convince himself still further that this is true. But what is here visually obvious from $m=2$ can be proved to be true for any finite value for m.

Consider any point $P=(a_1, \cdots, a_m)$ in the convex hull of $P_1, \cdots, P_n$ with reference to the point $S=(s_1, \cdots, s_m)$, S being the point in this same convex hull least distant from the point $0=(0, \cdots, 0)$. For every value for t between 0 and 1, we will have a point with coordinates $ta_i+(1-t)s_i$, $(i=1, \cdots, m)$, and this point will also be in the convex hull. In Diagram 10, for example, if $t=0.5$, we should have a point P'' (not shown in the diagram) falling midway between P and S, since

$$a_1'' = 0.5 \cdot 5/6 + (1 - 0.5) \cdot 1/2 = 2/3$$
$$a_2'' = 0.5 \cdot 5/6 + (1 - 0.5) \cdot 1/2 = 2/3,$$

so $P''=(2/3, 2/3)$, which is midway between $S=(1/2, 1/2)$ and $P=(5/6, 5/6)$.

P'' can clearly be no closer than is S to 0, and in fact P'' can be synonymous with S only if $t=0$. Consequently, we can say that

$$\sum_i [ta_i + (1 - t)s_i]^2 \geq \sum_i s_i^2$$

This inequality can, in turn, be expanded to

$$\sum_i [t^2a_i^2 + 2(ta_is_i - t^2a_is_i) - 2ts_i^2 + t^2s_i^2] \geq 0$$

and then rearranged and reduced to

$$2 \sum_i s_i(a_i - s_i) + \sum_i (a_i - s_i)_t{}^2 \geq 0$$

Consequently, as t tends to zero, the last inequality tends to

$$\sum_i s_i(a_i - s_i) \geq 0, \text{ or to}$$
$$\sum_i s_ia_i \geq \sum_i s_i^2 > 0$$

This proves what we set out to prove: namely, that, if the point $0=(0, \cdots, 0)$ does not belong to the convex hull $P_1, \cdots, P_n$, then values for $s_1, \cdots, s_m$ exist such that, for any P in the convex hull, we have

$$s_1a_1 + \cdots + s_ma_m > 0$$

(B) THEOREM OF THE ALTERNATIVE FOR MATRICES

Given, once more, values a_{ij} $(i=1, \cdots, m; j=1, \cdots, n)$. Consider both the convex hull formed by the set of n points in m-dimensional space and the further points $(1, 0, \cdots, 0)$, $(0, 1, 0, \cdots 0)$, $\cdots$, $(0, \cdots, 0, 1)$. Thus, in both Diagrams 10 and 11, we have 3 points in 2-dimensional space forming convex hulls, and the unit vectors (1, 0) and (0, 1) are points which also belong to those convex hulls. In Diagram 11, however, the point 0 belongs to the convex hull, whereas it does not belong to the convex hull in Diagram 10.

The theorem of the alternative for matrices says that:

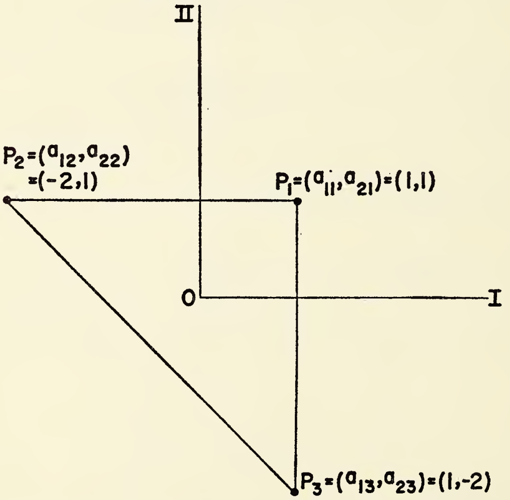
(1) if 0 and the unit vectors belong to the convex hull, there exists some y such that

(1a) $a_{i1}y_1 + \cdots + a_{in}y_n \leq 0$, for all i (i.e., for each *row* of a_{ij} coefficients)

(2) if 0 does not belong to the convex hull, but the unit vectors do belong to the convex hull, there exists some x such that

(2a) $a_{1j}x_1 + \cdots + a_{mj}x_m > 0$, for all j (i.e., for each *column* of a_{ij} coefficients)

DIAGRAM 11



As we shall shortly see, with the theorem of the supporting hyperplanes behind us, we shall have no difficulty proving (2a). But (1a) requires a little more effort.

To consider (1a), illustrating it at a small-dimensional level with

Diagram 11, if the point 0 belongs to $P_1, \cdots, P_n$, then there are values $t_1, \cdots, t_{n+m}$, all non-negative, such that $t_1 + \cdots + t_{n+m} = 1$, and such that

$$t_1a_{i1} + \cdots + t_na_{in} + t_{n+i} = 0, \text{ or,}$$
$$t_1a_{i1} + \cdots + t_na_{in} = -t_{n+i} \leq 0$$

In Diagram 11, for example, for $t = (2/18, 5/18, 5/18, 3/18, 3/18)$, we have

$$t_1a_{11} + \cdots + t_3a_{13} = -t_4$$
$$2/18(1) + 5/18(-2) + 5/18(1) = -3/18 < 0$$

or

$$t_1a_{21} + \cdots + t_3a_{23} = -t_5$$
$$2/18(1) + 5/18(1) + 5/18(-2) = -3/18 < 0$$

Alternatively, had we chosen $t_4 = t_5 = 0$ and computed t_1, t_2, and t_3 (as weights operating on the point 0 from P_1, P_2, and P_3) by the method we used in Diagram 10, we should have gotten $(t_1, t_2, t_3) = (1/2, 1/4, 1/4)$, and these weights would, of course, have given us the point (0, 0). Thus, in the former case we get that $t_1a_{i1} + \cdots + t_na_{in} = -t_{n+i} < 0$, while in the latter case we get that $t_1a_{i1} + \cdots + t_na_{in} = 0$. Neither of these things could happen, were the point 0 not in the convex hull.

Since $t_1 + \cdots + t_n$ is necessarily positive,[5] we can say that

$$y_j = \frac{t_j}{t_1 + \cdots + t_n}$$

satisfy (1a). Thus, in our first example, we have $y = (2/12, 5/12, 5/12)$ satisfying $a_{i1}y_1 + \cdots + a_{i3}y_3 \leq 0$, for all $i = 1, \cdots, 2$. This establishes the validity of (1a).

Now consider (2a), for which Diagram 10 is a small-dimension example. If the point 0 does not belong in the convex hull of $P_1, \cdots, P_n$, we know from the theorem of the supporting hyperplanes that there exist values for S such that

$$s_1a_{1j} + \cdots + s_ma_{mj} > 0, \text{ for all } j = 1, \cdots, n.$$

---

[5] If all t_i $(i = 1, \cdots, n+m)$, are non-negative and must add to unity, $t_1 + \cdots + t_n$ must be positive—otherwise, $t_1a_{i1} + \cdots \cdot t_na_{in} = -t_{n+i}$, only if $t_{n+i} = 0$. But this would, in itself, be a contradiction of assumptions, since then

$$\sum_{i=1}^{n+m} t_i = 0, \quad \text{rather than} \quad \sum_{i=1}^{n+m} t_i = 1.$$

Since the unit vectors, by hypothesis, belong in the convex hull, we also know that

$$s_1 > 0, \cdots, s_m > 0,$$

for this hypothesis places S in the all-positive-coordinates "quadrant." Consequently, we can say that

$$x_i = \frac{s_i}{s_1 + \cdots + s_m}$$

satisfy (2a). Thus, in our Diagram 10 example, not only $S=(1/2, 1/2)$ but $x=(1/2, 1/2)$ also satisfies $a_{1j}x_1+a_{2j}x_2>0$, for all $j=1, \cdots, 3$. This establishes the validity of (2a) and completes the proof of the theorem of the alternative for matrices.

(c) IMPOSSIBILITY THAT $\eta_1<0<\eta_2$

From the theorem of the alternative for matrices, if the point 0 belongs to the convex hull of $P_1, \cdots, P_n$, we have seen the validity of the statement that for some y

(1a) $$a_{i1}y_1 + \cdots + a_{in}y_n \leq 0, \text{ for all } i$$

Now the left side of (1a) can be expressed in the more compact form we used early in this chapter, and we can say that for some y

$$\sum a_{ij}y_j \leq 0, \text{ for all } i \text{ (i.e., for each matrix row)}$$

Consequently, it must also be true that

$$\max_i \sum a_{ij}y_j \leq 0$$

It necessarily follows, then, that we can also say that

$$\eta_2 = \min_y \max_i \sum_i \sum_j a_{ij}y_j \leq 0$$

Since we have proved earlier that $\eta_1 \leq \eta_2$, we can say that, if the point 0 is in the convex hull, it must follow that $\eta_1 \leq \eta_2 \leq 0$.

On the other hand, we also know from the theorem of the alternative for matrices that, if the point 0 is not in the convex hull (so S is in the all-positive-coordinates "quadrant"), it is valid to say that for some x

(2a) $$a_{1j}x_1 + \cdots + a_{mj}x_m > 0, \text{ for all } j$$

But the left side of (2a) can also be expressed more compactly to say that for some x

$$\sum_i a_{ij}x_i > 0, \text{ for all } j \text{ (i.e., for each matrix column)}$$

Consequently, it is also true that

$$\min_j \sum_i a_{ij}x_i > 0$$

Therefore, we know that

$$\eta_1 = \max_x \min_j \sum_i \sum_j a_{ij}x_i > 0$$

Again, knowing that $\eta_1 \leq \eta_2$, we can conclude that, if the point 0 is not in the convex hull (so S is in the all-positive-coordinates "quadrant"), it must follow that $\eta_2 \geq \eta_1 > 0$.

Now to say, from the conclusions drawn from (1a), that $\eta_1 \leq \eta_2 \leq 0$, and from the conclusions drawn from (2a), that $\eta_2 \geq \eta_1 > 0$, amounts to saying that it is impossible for $\eta_1 < 0$ at the same time that $\eta_2 > 0$. We have, then, that it is impossible for $\eta_1 < 0 < \eta_2$. But this still leaves open the possibility of $\eta_1 < \eta_2$, when both η_1 and η_2 are simultaneously either smaller than or greater than 0.

(D) IMPOSSIBILITY THAT $\eta_1 < k < \eta_2$

Suppose η_1 (the maximin) is less than 0 by k, where k is any number. Further suppose that we add to each a_{ij} element of our matrix an amount infinitesimally smaller than k, call it $\hat{k}$.[6] Addition of such a constant obviously does not alter the *solution* to the game, but it does have the effect of increasing the

$$\sum_j a_{ij}y_j, \text{ for all } i,$$

and the

$$\sum_i a_{ij}x_i, \text{ for all } j, \text{ by } \hat{k}.$$

From this, it follows that both η_1 and η_2 increase by $\hat{k}$, and the value of η_1 is now infinitesimally smaller than 0.[7] But we know that η_2 cannot be greater than 0, if η_1 is smaller than 0. Therefore, η_1 could have been no more than infinitesimally smaller than η_2, before $\hat{k}$ was added to the a_{ij} elements of the matrix. Evidently, then, we can conclude that it is also true that it is impossible for $\eta_1 < k$ at the same time that $\eta_2 > k$. Knowing that $\eta_1 \leq \eta_2$, it follows that it must be true that always $\eta_1 = \eta_2$. This is precisely the statement of the minimax theorem we set out to prove.

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[6] If η_1 were *greater* than zero by k, we could, of course, *subtract* from each a_{ij} element an amount infinitesimally *larger* than k to arrive at the same conclusion reached in this and the two immediately following paragraphs.

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[7] This can be easily visualized with reference to Diagram 11. The solution to this problem is $y = (0, 1/2, 1/2)$, strategy No. 1 of B being dominated by B's other two strategies, and $x = (1/2, 1/2)$. The value of the game is $\eta_1 = \eta_2 = -1/2$. If we add infinitesimally less than 1/2 to each of the elements of this 2×3 matrix, the convex hull of $P_1P_2P_3$ is merely shifted up and to the right, until the new line joining P_2 and P_3 falls infinitesimally short of passing through the point 0, or the origin. The solution remains unaltered, but the value of the game rises by $\hat{k}$.

CHAPTER VIII

GAMES: THEIR DUALITY AND THEIR SOLUTION

It was suggested in the preceding chapter that solving games having no saddle point was frequently difficult. It was also suggested that solving two-person zero-sum games involved the solution of dual sets of equations. In this chapter we shall be concerned with the related problems of understanding this duality nature of games and of seeing how this makes it possible to solve such games by turning them into linear programming problems. The final game considered in this chapter will contain the hint that a linear programming problem can also be turned into a game, and this possibility will be examined in Chapter IX.

THE DUALITY OF GAMES

It is clear from what we learned in the preceding chapter that, given a pay-off matrix and the choice of a mixed strategy x by A, A will gain not less than

$$\min_j \sum_i a_{ij}x_i = \eta.$$

This being so, we can say that

$$a_{1j}x_1 + \cdots + a_{mj}x_m \geq \eta, \text{ for all } j = 1, \cdots, n$$

$$x_1 + \cdots + x_m = 1, \text{ all } \overset{m}{\underset{i=1}{x_i}} \geq 0$$

While η, which A naturally wants to maximize, may not be positive, we know we can make $\eta > 0$ by simply adding an appropriate constant to all a_{ij} elements, without altering the *solution* (but not the value of the game) in any way. With η positive, we can divide each of the above $(n+1)$ equations through by η to get

$$a_{1j}\frac{x_1}{\eta} + \cdots + a_{mj}\frac{x_m}{\eta} \geq 1, \text{ for all } j = 1, \cdots, n$$

$$\frac{x_1 + \cdots + x_m}{\eta} = \frac{1}{\eta}, \text{ all } \overset{m}{\underset{i=1}{x_i}} \geq 0$$

If we now define new variables $x_i' = x_i/\eta$, our system becomes

$$\sum_{i=1}^{m} a_{ij}x_i' \geq 1, \text{ for all } j = 1, \cdots, n$$

$$\text{Minimize } \sum_{i=1}^{m} x_i' = \frac{1}{\eta} \text{ (because } A \text{ seeks to maximize } \eta)$$

Similarly, we should get for B the following system

$$\sum_{j=1}^{n} a_{ij}y_j' \leq 1, \text{ for all } i = 1, \cdots, m$$

$$\text{Maximize } \sum_{j=1}^{n} y_j' = \frac{1}{\eta} \text{ (because } B \text{ seeks to minimize } \eta)$$

Near the end of Chapter VI we saw that dual sets of inequalities and their functionals could be stated

$$\sum_{i=1}^{m} \alpha_{ij}x_i \geq \gamma_j \qquad\qquad \sum_{j=1}^{n'} \alpha_{ij}y_j \leq \beta_i$$

$$\text{Minimize } \sum_{i} \beta_i x_i \qquad\qquad \text{Maximize } \sum_{j} \gamma_j y_j$$

These latter two systems from Chapter VI are clearly a more general statement than that represented by the two above systems for games. They are more general, since the right sides of the requirements inequalities have the more general coefficients of β_i and γ_j (rather than of unity), and since the functionals have the more general coefficients of γ_j and β_i (rather than of unity). Thus the games problem is actually a special case of the more general case considered in Chapter VI.

SOLUTION OF GAMES

Knowing that a simplex tableau implicitly deals simultaneously with a pair of dual problems, and knowing also that a two-person zero-sum game is a special case of this sort, it is clear that we should be able to solve A's and B's problems simultaneously (as well as finding the value of the game), by simply formulating the problem of one of the players as a linear programming problem and solving it.

Consider the game represented by the following payoff matrix:

		B		A's probabilities
	−1	0	1	1/3
A	0	1	−1	1/3
	1	−1	0	1/3
B's probabilities	1/3	1/3	1/3	

The value of this game is 0, and the correct solution to this game is one in which both *A* and *B* assign equal weights of 1/3 to each of their three pure strategies. We can readily verify this by turning the game into a linear programming problem to minimize *B*'s loss.

From the preceding chapter, we know that, if *B* chooses a particular mixed strategy *y*, he can hold his loss down at least as low as

$$\max_{i} \sum_{j} a_{ij}y_j = \eta.$$

Consequently, we know that

$$a_{i}y_1 + \cdots a_{in}y_n \leq \eta, \text{ for all } i = 1, \cdots, m$$

We can turn this system of inequalities into a linear programming problem in either of two general ways. One way is to view the problem, as suggested in the preceding section, as one of maximizing

$$\sum_{j=1}^{n} y_j' = \frac{1}{\eta};$$

and this is a case of minimizing η indirectly. The other way is to view the problem as one of minimizing η directly.

We cannot use the former approach of maximizing

$$\sum_{j=1}^{n} y_j' = \frac{1}{\eta},$$

unless $\eta > 0$; but we can assure ourselves that $\eta > 0$, by adding 2 (in fact, adding any number ≥ 1 will obviously work, in this case) to each of the elements of the game matrix. If we do this, then dividing the three resulting new inequalities through by η, and if we finally add one additional variable to each of these inequalities to turn them into equations, we get:

$$\begin{array}{rcl}
y_1' + 2y_2' + 3y_3' + y_4 \qquad\qquad & = & 1 \\
2y_1' + 3y_2' + y_3' \qquad + y_5 \qquad & = & 1 \\
3y_1' + y_2' + 2y_3' \qquad\qquad + y_6 & = & 1 \\
\text{Maximize } y_1' + y_2' + y_3' & & \left(= \dfrac{1}{\eta}\right)
\end{array}$$

The optimum solution to this problem is in terms of $y' = (y_1', y_2', y_3')$ and looks as follows:

$$\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} -5/18 & 1/18 & 7/18 \\ 1/18 & 7/18 & -5/18 \\ 7/18 & -5/18 & 1/18 \end{bmatrix}$$

$$\cdot\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \overset{0}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} y_4 - \overset{0}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} y_5 - \overset{0}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} y_6\right\}$$

$$= \begin{bmatrix} 1/6 \\ 1/6 \\ 1/6 \end{bmatrix} - \begin{bmatrix} -5/18 \\ 1/18 \\ 7/18 \end{bmatrix} y_4$$

$$z_{00} = \frac{1}{\eta} = 1/2 \qquad z_{04} = -x_4' = -1/6$$

$$- \begin{bmatrix} 1/18 \\ 7/18 \\ -5/18 \end{bmatrix} y_5 - \begin{bmatrix} 7/18 \\ -5/18 \\ 1/18 \end{bmatrix} y_6$$

$$z_{05} = -x_5' = -1/6 \qquad z_{06} = -x_6' = -1/6$$

All $z_{0,m+k}$ being negative, we know we have an optimum maximization solution.

Since $z_{00} = 1/\eta = 1/2$, we know that the value of the *new* game (i.e., the game posed by adding $k = 2$ to each element of the original matrix) is 2. This tells us that the value of the *original* game is $(2-2)=0$. Furthermore, since

$$(y_1', y_2', y_3') = \left(\frac{y_1}{\eta}, \frac{y_2}{\eta}, \frac{y_3}{\eta}\right) = (1/6, 1/6, 1/6),$$

we know that $(y_1, y_2, y_3)=(1/3, 1/3, 1/3)$, i.e., that an optimum mixed strategy for B is to play each of his three strategies 1/3 of the time. Finally, since we know (from Chapter VI) that $p_{m+k,0}=-z_{0,m+k}$, we also know that

$$(x_4', x_5', x_6') = \left(\frac{x_4}{\eta}, \frac{x_5}{\eta}, \frac{x_6}{\eta}\right) = (1/6, 1/6, 1/6),$$

so that $(x_4, x_5, x_6)=1/3, 1/3, 1/3)$ for A's three strategies.

On the other hand, if we choose to view the game problem as one of minimizing η directly, the m inequalities implicit in the *original* game matrix can immediately be turned into equations by adding one new variable,

$$\underset{i=1}{\overset{m}{y_{n+i}}},$$

to the left side of each of the m inequalities. Doing this, and having also that $y_1+\cdots+y_n=1$, with $y_1, \cdots, y_n \geq 0$, along with the knowledge that B wants to minimize η,[1] we can formulate B's problem as:

$$\begin{array}{rrrrrrl}
y_1 & + y_2 & + y_3 & & & & = 1 \\
-y_1 & & + y_3 & + y_4 & & & = \eta \\
& y_2 & - y_3 & & + y_5 & & = \eta \\
y_1 & - y_2 & & & & + y_6 & = \eta
\end{array}$$

Minimize η for non-negative variables

We can rid this system of η, at the same time reducing the number of equations by one, if we subtract the second and third requirements equations from the last requirements equation,[2] and if we specify that we wish to minimize the left side of this last equation (which $=\eta$). B's problem then becomes the following typical linear programming problem:

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[1] η is not, of course, a constant, but changes with changing solutions resulting from changes in $y=(y_1, \cdots, y_n)$ as we move towards the optimum solution.

[2] Note that these last three requirements equations are so ordered that the coefficient in the first column of each equation is smaller than the same coefficient of the next lower equation. This results in our being sure that, after the subtraction process, all of the coefficients in the first column will be non-negative. The importance of this will shortly become clear.

$$\begin{aligned} y_1 + y_2 + y_3 &= 1 \\ 2y_1 - y_2 - y_3 - y_4 + y_6 &= 0 \\ y_1 - 2y_2 + y_3 - y_5 + y_6 &= 0 \end{aligned}$$

$$\text{Minimize } y_1 - y_2 + y_6$$

As we solve this problem, we know that we shall also be solving A's dual set of equations, so that we shall emerge simultaneously with a solution to both A's and B's problems.

It actually takes a rather rare 3×3 matrix to get a game in which all three of the pure strategies of both opponents are active. There is, therefore, no reason to assume that B's loss will be minimized by choosing y_1, y_2, and y_3 as our basic variables—even though this is, in this case, true.

Suppose, then, that we choose y_1, y_4, and y_5 as our initial basic variables. The matrix of coefficients for these variables is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

The left column of this matrix is completely non-negative, since we ordered our requirements equations so as to achieve this end. Furthermore, our choice of y_4 and y_5 as the other two basic variables results in the remaining six elements of the matrix being such that $A=A^{-1}$.

To choose y_1, y_4, and y_5 as our initial basic variables is, then, eminently sensible. For we know immediately what our inverse matrix is, and need not bother to calculate it. Since the right side of our three equations is the vector

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

we also know that the left column of $A = A^{-1}$ represents the values of y_1, y_4, and y_5 in the first-stage solution.

Indeed, the first-stage solution is

$$\begin{matrix} c_1 = 1 \\ c_4 = 0 \\ c_5 = 0 \end{matrix} \begin{bmatrix} y_1 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\cdot \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \overset{c_2 = 1}{\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}} y_2 - \overset{c_3 = 0}{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}} y_3 - \overset{c_6 = 1}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}} y_6 \right\}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 3^* \end{bmatrix} y_2 - \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} y_3 - \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} y_6$$

$$\not{p}_{00} = 1 \quad \not{p}_{02} = x_2 = -2 \quad \not{p}_{03} = x_3 = -1 \quad \not{p}_{06} = x_6 = 1$$

If we now choose to bring y_2 into the basis, our pivot becomes $\not{p}_{52} = 3$, so we remove y_5 in the second-stage solution ultimately to get (with the help of our algorithms from the appendix of Chapter III)

$$\begin{matrix} c_1 = & 1 \\ c_2 = & -1 \\ c_4 = & 0 \end{matrix} \begin{bmatrix} y_1 \\ y_2 \\ y_4 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ 1 \end{bmatrix} - \overset{c_3 = 0}{\begin{bmatrix} 1 \\ 0 \\ 3^* \end{bmatrix}} y_3$$

$$\not{p}_{00} = 1/3 \quad \not{p}_{03} = x_3 = -1$$

$$- \overset{c_5 = 0}{\begin{bmatrix} -1/3 \\ 1/3 \\ -1 \end{bmatrix}} y_5 - \overset{c_6 = 0}{\begin{bmatrix} 1/3 \\ -1/3 \\ 0 \end{bmatrix}} y_6$$

$$\not{p}_{05} = x_5 = 2/3 \qquad \not{p}_{06} = x_6 = 1/3$$

This leaves us with only one choice as to the variable next to be introduced into the basis, and our pivot becomes $\not{p}_{43}$, with y_4 the variable to be removed. This leads to the final solution of

$$\begin{array}{r} c_1 = 1 \\ c_2 = -1 \\ c_3 = 0 \end{array} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \underset{\not{p}_{00} = z_{00} = 0}{\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}} - \overset{c_4 = 0}{\underset{\not{p}_{04} = x_4 = 1/3}{\begin{bmatrix} 0 \\ 1/3 \\ -1/3 \end{bmatrix}}} y_4$$

$$- \overset{c_5 = 0}{\underset{\not{p}_{05} = x_5 = 1/3}{\begin{bmatrix} -1/3 \\ 0 \\ 1/3 \end{bmatrix}}} y_5 - \overset{c_6 = 1}{\underset{\not{p}_{06} = x_6 = 1/3}{\begin{bmatrix} 1/3 \\ -1/3 \\ 0 \end{bmatrix}}} y_6$$

We find from $\not{p}_{00} = z_{00}$ that the value of the game is zero. And we know this is a stable value, because the signs of the remaining three $\not{p}$'s tell us that $\not{p}_{00}$ cannot be lowered further. Finally, since $\not{p}_{0,m+k} = z_{m+k,0} = x_{m+k,0}$, the values of these three $\not{p}$'s[3] represent the probability values to be assigned to *A*'s three pure strategies to maximize *A*'s gains; and the values for y_1, y_2, and y_3 are, of course, *B*'s probability values for his three pure strategies.

LINEAR PROGRAMMING PROBLEMS AS GAMES

The duality results obtained in Chapter VI lead also to the converse of the proposition we have just considered: not only is it possible to solve a game by turning it into a linear programming problem, it is also possible to turn a linear programming problem into a game, then solving the linear programming problem by any known method of solving a game. The primary value of this knowledge is that it places at one's disposal, in solving linear programming problems, all the theorems involved in game theory, in such instances as these theorems would facilitate the solution of a linear programming problem.

Turning a linear programming problem into a game, this game then to be solved by any known method of solving a game, introduces certain considerations new to us. These considerations will stand out more clearly if we break our development of them into three parts.

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[3] Because we are now deducing the Max-D from the Min-D, rather than the reverse, rule (3) on page 74 of Chapter VI changes from $\not{p}_{0u} = z_{u0}$ to $\not{p}_{0,m+k} = z_{m+k,0}$. Similarly, rule (2) would be supplanted by $z_{u0} = -\not{p}_{0u}$.

First, we shall take the matrix of coefficients from the blending problem of Chapters I and VI, turn this matrix into a 4×4 game matrix, and then solve it as a game. This will be a more complicated game than that of the preceding section of this chapter; but, methodologically speaking, we shall learn nothing new about solving a game as a linear programming problem. Nevertheless, we should gain from this procedure, for we shall have laid a foundation for more easily acquiring an insight into the process of turning a linear programming problem into a game.

Second, because the preceding step will also lead us to the conclusion that solving the above 4×4 game matrix really does not constitute a solution to the linear programming problem, we shall consider (in Chapter IX) what is necessary to turn the linear programming problem into a game such that a solution of that game actually does constitute a solution to the linear programming problem as well. In effect, this will amount to considering the steps necessary to remedy the shortcomings of our 4×4 game-matrix approach.

Finally, although a game *always* has a solution, a linear programming problem may sometimes have either no optimum solution or an unbounded set of optimizing x's or y's, even if there is an optimum solution. We shall see (also in Chapter IX) that attempting to solve a linear programming problem as a game will ultimately reveal to us that either of these situations exists, if such is in fact the case.

The 4×4 game matrix. Consider the maximization-problem inequalities and their functional on page 67 of Chapter VI. If we substitute the optimizing value of 2,500 for Z in the functional, we have three inequations and an equation. Now let us move the right side of each of these to the left side, changing the signs, of course, of these four coefficients. Then let us change *all* the signs of the *functional* coefficients, which naturally leaves the validity of the functional unchanged. Finally, let us change the equality sign of the functional equation to an *in*equality sign of $\leq$, even though we know the $<$ part of this sign is superfluous. We now have four inequations, with the right sides all reading ≤ 0, and the left sides have a total of 16 coefficients (although 3 of the 16 are zeros, and there is no variable explicitly associated with the fourth column of coefficients).

We can look at these 16 coefficients as a 4×4 game matrix. The game then looks as follows, with its solution in terms of A's and B's probabilities also given:

	B				A's probabilities
A	2	1	−2	0	$\frac{4,503}{15,010}$
	1	1	−3	0	0
	1	1	0	−1,000	$\frac{7,505}{15,010}$
	−5	−4	3	2,500	$\frac{3,002}{15,010}$
B's probabilities	0	$\frac{1,000}{1,501}$	$\frac{500}{1,501}$	$\frac{1}{1,501}$	

Since *A*'s second row is dominated by *A*'s first row, in solving this problem we could eliminate the second row. But we prefer to do this the hard way, turning the full 4×4 game into a linear programming problem of minimizing *B*'s losses.

We know the problem can thus be stated as:

$$\begin{aligned}
Y_1 + Y_2 + Y_3 + Y_4 &= 1\\
2Y_1 + Y_2 - 2Y_3 + 0Y_4 + Y_5 &= \eta\\
Y_1 + Y_2 - 3Y_3 + 0Y_4 + Y_6 &= \eta\\
Y_1 + Y_2 + 0Y_3 - 1{,}000Y_4 + Y_7 &= \eta\\
-5Y_1 - 4Y_2 + 3Y_3 + 2{,}500Y_4 + Y_8 &= \eta
\end{aligned}$$

Subtracting the second, third, and fourth equations, successively, from the last equation, we get

$$\begin{aligned}
Y_1 + Y_2 + Y_3 + Y_4 &= 1\\
-7Y_1 - 5Y_2 + 5Y_3 + 2{,}500Y_4 - Y_5 + Y_8 &= 0\\
-6Y_1 - 5Y_2 + 6Y_3 + 2{,}500Y_4 - Y_6 + Y_8 &= 0\\
-6Y_1 - 5Y_2 + 3Y_3 + 3{,}500Y_4 - Y_7 + Y_8 &= 0
\end{aligned}$$

$$\text{Minimize } -5Y_1 - 4Y_2 + 3Y_3 + 2{,}500Y_4 + Y_8$$

$$(= \eta, \text{ which } B \text{ wants to minimize})$$

Even this relatively small 4×4 game is a tedious one to solve. Indeed this is still true, should we initially guess that the proper variables for which to solve are Y_2, Y_3, Y_4, and Y_6. This is evident when we see the cumbersome inverse matrix in relation to the problem:

$$\begin{array}{r} -4 \\ 3 \\ 2{,}500 \\ 0 \end{array} \begin{bmatrix} Y_2 \\ Y_3 \\ Y_4 \\ Y_6 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} \frac{1{,}000}{1{,}501} & -\frac{3{,}497}{15{,}010} & 0 & \frac{499}{3{,}002} \\ \frac{500}{1{,}501} & \frac{3{,}505}{15{,}010} & 0 & -\frac{501}{3{,}002} \\ \frac{1}{1{,}501} & -\frac{8}{15{,}010} & 0 & \frac{2}{3{,}002} \\ \frac{500}{1{,}501} & \frac{18{,}515}{15{,}010} & -1 & -\frac{501}{3{,}002} \end{bmatrix} \\ \begin{array}{cccc} -5 & \quad\quad 0 & \quad\quad 0 & \quad\quad 1 \end{array} \end{array}$$

$$\cdot \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ -7 \\ -6 \\ -6 \end{bmatrix} Y_1 - \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} Y_5 - \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} Y_7 - \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} Y_8 \right\}$$

$$= \begin{bmatrix} \frac{1{,}000}{1{,}501} \\ \frac{500}{1{,}501} \\ \frac{1}{1{,}501} \\ \frac{500}{1{,}501} \end{bmatrix} - \overset{-5}{\begin{bmatrix} \frac{19{,}509}{15{,}010} \\ -\frac{4{,}505}{15{,}010} \\ \frac{6}{15{,}010} \\ -\frac{19{,}515}{15{,}010} \end{bmatrix}} Y_1 - \overset{0}{\begin{bmatrix} \frac{3{,}497}{15{,}010} \\ -\frac{3{,}505}{15{,}010} \\ \frac{8}{15{,}010} \\ -\frac{18{,}515}{15{,}010} \end{bmatrix}} Y_5$$

$$z_{00} = 0 \qquad z_{01} = 1/10 \qquad z_{05} = \frac{4{,}503}{15{,}010}$$

$$\begin{array}{cccc} & 0 & & 1 \\ - & \begin{bmatrix} -\dfrac{499}{3{,}002} \\ \dfrac{501}{3{,}002} \\ -\dfrac{2}{3{,}002} \\ \dfrac{501}{3{,}002} \end{bmatrix} Y_7 & - & \begin{bmatrix} -\dfrac{1{,}002}{15{,}010} \\ \dfrac{1{,}000}{15{,}010} \\ \dfrac{2}{15{,}010} \\ \dfrac{16{,}010}{15{,}010} \end{bmatrix} Y_8 \\ & z_{07} = 1/2 & & z_{08} = 1/5 \end{array}$$

All the z's shown at the bottom being positive, we know we have an optimum solution.

Since Y_6 is not one of B's strategies, but $Y_1(=0)$ is, we get the Y_1, Y_2, Y_3, and Y_4 probabilities indicated for B. Similarly, from the fundamental duality theorem we know that $z_{01}=X_5$, $z_{05}=X_1$, $z_{07}=X_3$, and $z_{08}=X_4$. But X_5 is not one of A's strategies, although X_2 is. At the same time, however, we know that Y_6 is X_2 from A's (i.e., the dual's) viewpoint. Consequently, since Y_6 (or X_2) is a *non*-basic variable from A's viewpoint, $Y_6=X_2=0$. Thus we get the X_1, X_2, X_3, and X_4 probabilities indicated for A.

It is clear that the probability figures thus given for A and B constitute a solution of our 4×4 *game.* On the other hand, we have *not* solved the *linear programming problem* of Chapters I and VI. If we carefully examine our optimum solution in the form of Y_2, Y_3, Y_4, and Y_6, however, we can make an interesting discovery.

Now Y_4 is, of course, the variable associated with the fourth column of our 4×4 matrix, and

$$Y_4 = \frac{1}{1{,}501} \cdot$$

If we divide Y_2, Y_3, and Y_6 by

$$Y_4 = \frac{1}{1{,}501},$$

we get $Y_2'=1{,}000$, $Y_3'=500$, and $Y_6'=500$. These *are* the values for the y_1, y_2, and y_3 solution to the linear programming problem on page 72 of Chapter VI. In brief, these *are* the values for visbreaker feed, flux, and gravity giveaway.

In essence, this means that $Y_2 = Y_4 y_1$, $Y_3 = Y_4 y_2$, and $Y_6 = Y_4 y_3$. Similarly, if we should look at the *min*imization dual of Chapter VI, we should discover that the X_1, X_3, and X_5 of our current 4×4 game are, respectively, since $z_{08} = X_4$, actually $X_4 x_5$, $X_4 x_6$, and $X_4 x_4$. In other words, we can also get the solution to the minimization dual of Chapter VI, by dividing X_1, X_3, and X_5 by X_4. Note, however, that this is possible only because both Y_4 and X_4 are positive—an important point to be considered in the next chapter.

Nevertheless, it must be emphasized that we could never have solved our true linear programming problem by the method we have just used. To "solve" the linear programming problem in the manner we just have, we had to cheat. For, by placing the a_{44} element equal to 2,500, we were putting what we knew to be the optimum solution into the matrix. Without knowing that part of the solution, we could not have solved our 4×4 matrix to get Y_2, Y_3, Y_4, and Y_6, and hence to get y_1, y_2, and y_3. It is obviously not cricket to "solve" a problem by starting with inside knowledge about part of the solution. Fortunately, however, there is a legitimate way to get the equivalent of Y_4 and X_4 (and hence of y_1, y_2, y_3, x_4, x_5, and x_6) by first turning the linear programming problem into a game, and we are now ready to consider this method in Chapter IX.

CHAPTER IX

LINEAR PROGRAMMING PROBLEMS AS GAMES

At the end of Chapter VIII, it was concluded that the solution to the 4×4 game matrix considered in the last section of that chapter did not offer a true solution to the linear programming problem of Chapter VI, from which the 4×4 game matrix was drawn. Anticipating that conclusion, it was stated, early in the last section of Chapter VIII, that Chapter IX would need to treat two matters: (1) the matter of how to turn a linear programming problem into a game in such a way that the solution to the game would also provide us with a true solution to the linear programming problem, and (2) the matter of how to recognize, from the game solution, either the possible existence of no optimum solution to the linear programming problem or the possibility of an unbounded set of optimizing x's or y's, even if an optimum solution does exist. These two matters are treated in the above order in the two sections of this chapter.

THE SKEW-SYMMETRIC MATRIX

The *initial* maximization problem on page 67 of Chapter VI, using the y subscripts *finally* used on page 69 of Chapter VI, was:

$$\begin{aligned} 2y_4 + y_1 - 2y_2 &\leq 0 \\ y_4 + y_1 - 3y_2 &\leq 0 \\ y_4 + y_1 \qquad\quad &\leq 1{,}000 \\ \text{Maximize } 5y_4 + 4y_1 - 3y_2 &= Z \end{aligned}$$

These haphazard-appearing subscripts, as we know, were purposely used in Chapter VI, because they resulted in the basic variables of the optimum solution being named y_1, y_2, and y_3 and hence in the nonbasic variables being neatly labeled as y_{m+k} (with $m=3$ and $k=1, \cdots, 3$).

The minimization problem, as we also know from Chapter VI, can be expressed (also using the *final* x subscripts of the *initial* problem of Chapter VI) as:

$$\begin{aligned} 2x_7 + \ x_8 + x_9 &\geq 5 \\ x_7 + \ x_8 + x_9 &\geq 4 \\ -2x_7 - 3x_8 \qquad &\geq -3 \\ \text{Minimize } 1{,}000x_9 &= \not{Z} \end{aligned}$$

If we move the right sides of both sets of these inequalities to the left sides of the inequalities, and if we change all the signs of the second set of inequalities in order to change the second set's inequality sign from $\geq$ to $\leq$, we get:

$$\begin{array}{lcr}
 & 2y_4 + y_1 - 2y_2 + & 0 \leq 0 \\
 & y_4 + y_1 - 3y_2 + & 0 \leq 0 \\
 & y_4 + y_1 \qquad\qquad & -\ 1{,}000 \leq 0 \\
-2x_7 - x_8 - x_9 & & +\quad 5 \leq 0 \\
-x_7 - x_8 - x_9 & & +\quad 4 \leq 0 \\
2x_7 + 3x_8 & & -\quad 3 \leq 0 \\
\textit{Minimize } \not{Z} = 1{,}000x_9 & \multicolumn{2}{r}{\textit{Maximize } Z = 5y_4 + 4y_1 - 3y_2}
\end{array}$$

The column of coefficients just to the left of the inequality signs has no unknown variable, but is a column of bare coefficients. To turn the combined problem into a game problem, we must remedy this deficiency by multiplying the last column by such an unknown (say) z. But we cannot do this without distorting our problem unless we multiply all of the matrix elements by z.

At the same time that we do this, let us do one other thing. We know that when we reach either an optimum solution to a linear programming problem or a solution to a game, $Z = \not{Z}$. Therefore, we can subtract Z from $\not{Z}$ to get $\not{Z} - Z \leq 0$, although we know that the $<$ sign is superfluous. Thus, even though we do not know the value for Z (as we assumed we did in our 4×4 game matrix), we can formulate a row analogous to the fourth row of the 4×4 matrix, since we do know that $Z = \not{Z}$.

Doing these two things, and filling in the "holes" with zero-coefficient variables, we emerge with a system of seven inequalities:

$$\begin{array}{rrrrrrrr}
 & 0x_7z + 0x_8z + & 0x_9z + & 2y_4z + & y_1z - & 2y_2z + & 0z \leq 0 \\
 & 0x_7z + 0x_8z + & 0x_9z + & y_4z + & y_1z - & 3y_2z + & 0z \leq 0 \\
 & 0x_7z + 0x_8z + & 0x_9z + & y_4z + & y_1z + & 0y_2z - & 1{,}000z \leq 0 \\
 & -2x_7z - x_8z - & x_9z + & 0y_4z + & 0y_1z + & 0y_2z + & 5z \leq 0 \\
 & -x_7z - x_8z - & x_9z + & 0y_4z + & 0y_1z + & 0y_2z + & 4z \leq 0 \\
 & 2x_7z + 3x_8z + & 0x_9z + & 0y_4z + & 0y_1z + & 0y_2z - & 3z \leq 0 \\
\not{Z} - Z = & 0x_7z + 0x_8z + & 1{,}000x_9z - & 5y_4z - & 4y_1z + & 3y_2z + & 0z \leq 0
\end{array}$$

Finally, we may let:

$$W_1 = x_7z;\ W_2 = x_8z;\ W_3 = x_9z;\ W_4 = y_4z;\ W_5 = y_1z;\ W_6 = y_2z;\ W_7 = z$$

and add the restraint that

$$\sum_{j=1}^{7} W_j = 1,$$

with each W_j non-negative. This gives us:

$$\begin{aligned}
0W_1 + 0W_2 + {}& 0W_3 + 2W_4 + 1W_5 - 2W_6 + 0W_7 \leq 0 \\
0W_1 + 0W_2 + {}& 0W_3 + 1W_4 + 1W_5 - 3W_6 + 0W_7 \leq 0 \\
0W_1 + 0W_2 + {}& 0W_3 + 1W_4 + 1W_5 + 0W_6 - 1{,}000W_7 \leq 0 \\
-2W_1 - 1W_2 - {}& 1W_3 + 0W_4 + 0W_5 + 0W_6 + 5W_7 \leq 0 \\
-1W_1 - 1W_2 - {}& 1W_3 + 0W_4 + 0W_5 + 0W_6 + 4W_7 \leq 0 \\
2W_1 + 3W_2 + {}& 0W_3 + 0W_4 + 0W_5 + 0W_6 - 3W_7 \leq 0 \\
0W_1 + 0W_2 + {}& 1{,}000W_3 - 5W_4 - 4W_5 + 3W_6 + 0W_7 \leq 0
\end{aligned}$$

This system can clearly be treated as a game for which $\eta \leq 0$, from *B*'s viewpoint. Setting the coefficients in the form of a game matrix, we have:

	B							*A*'s probabilities:
	0	0	0	2	1	−2	0	$\frac{3}{3{,}010}$
	0	0	0	1	1	−3	0	0
	0	0	0	1	1	0	−1,000	$\frac{5}{3{,}010}$
A	−2	−1	−1	0	0	0	5	0
	−1	−1	−1	0	0	0	4	$\frac{2{,}000}{3{,}010}$
	2	3	0	0	0	0	−3	$\frac{1{,}000}{3{,}010}$
	0	0	1,000	−5	−4	3	0	$\frac{2}{3{,}010}$
B's probabilities:	$\frac{3}{3{,}010}$	0	$\frac{5}{3{,}010}$	0	$\frac{2{,}000}{3{,}010}$	$\frac{1{,}000}{3{,}010}$	$\frac{2}{3{,}010}$	

This table of coefficients constitutes, in matrix algebra terminology, a skew-symmetric matrix—a square matrix in which the first column is the same as the first row with signs reversed, and so on for the successive n columns and rows. Being square, this matrix has a determinant, and the value of the determinant of a skew-symmetric matrix is always zero. So, by combining the maximization and minimization problems of Chapter VI into a single problem, we have constructed a game the value of which must be zero. And we have done this without having to know the optimum value for the functionals from which the bottom row of the matrix was derived.

We can now solve this game by any known method. From the solution to the game, we can then derive the solution to the linear programming problem. If we solve this game, we get the solution already indicated by A's and B's probabilities. A and B are not, of course, the A and B of the 4×4 matrix. The A and B of the larger matrix are essentially the same "person"—that is, the dual of a skew-symmetric matrix is itself, so the first (second, etc.) row and first (second, etc.) column probabilities are necessarily identical.

Having once solved the game and hence knowing that

$$W_7 = z = \frac{2}{3{,}010},$$

we can then divide W_1 to W_6 inclusive by z to get $x_7 = 3/2$; $x_8 = 0$; $x_9 = 5/2$; $y_4 = 0$; $y_1 = 1{,}000$; $y_2 = 500$. And with these values now obtained, we can readily obtain the optimum value for the maximization and minimization functionals.

In this particular problem, despite the lack of sign restrictions on x_7, x_8, and x_9, x_7 and x_9 are the equivalent of x_5 and x_6 in the minimization problem of Chapter VI, as can be readily verified by referring to page 71 of Chapter VI.

Now if we multiply the successive coefficients of the second and fourth rows (columns) of the game matrix by B's (A's) probabilities, we get figures $<(>)$ zero. This means that our second and fourth inequalities can only be turned into equations with a zero value on the right side by the inclusion of one additional variable of positive value for each of these two equations. Call these two variables W_9 and W_{11}.

W_9 will have a value of $\dfrac{1{,}000}{3{,}010}$, while W_{11} will have a value of $\frac{1}{3{,}010}$.

W_9 and W_{11} are, of course, y_3z and x_4z. Dividing these by

$$W_7 = z = \frac{2}{3{,}010},$$

we get $y_3 = 500$ and $x_4 = 1/2$. As we know, these are the correct values for the y_3 and x_4 of the duality problem of Chapter VI.

In point of fact, to solve the linear programming problem of Chapters I and VI by first turning it into a game is more onerous than to handle it as a straightforward linear programming problem. But it is easy to construct a problem for which this would not be so, and we have used the blending problem only because it was a familiar one, which made the relevant comparisons easier to assimilate.

THE CASE OF $z=0$

In the 4×4 and 7×7 matrices of Chapter VIII and the preceding section of this chapter, it was found that Y_4 and $W_7(=z)$ were both *positive*. Since negative values for all W_j are excluded by postulate, z cannot be less than zero. The postulate does not, however, exclude the possibility of $z=0$. The question naturally arises, therefore, as to what the significance may be of a zero value for z, should such a situation arise in the solution of a linear programming problem turned into a game.

Any solution to such a game which involves $z=0$ is clearly no solution to the linear programming problem from which that game was derived. This follows because the

$$\underset{i=1}{\overset{m}{x_i}} \text{ and } \underset{j=1}{\overset{n}{y_j}}$$

of the linear programming problem can only be found by dividing each of the W_{m+n} by $z = W_{m+n+1}$, and division by zero is meaningless. Indeed, if a solution to such a game does entail $z = W_{m+n+1} = 0$, and if that solution is an optimal strategy for the game, one of two things is true of the original linear programming problem: (1) either the problem is one without a finite maximum (minimum) solution for the *functional*, or (2) there is a finite maximum (minimum) solution for that functional, but *the set of x's or y's* involved in this optimal solution is unbounded. Let us consider an example of each of these two possibilities, in order to make these clearer.

No finite solution. A case involving no finite minimum is provided by:

$$\begin{aligned} 2x_1 - x_2 &\geq -2 \\ -x_1 + 2x_2 &\geq -2 \\ x_1 + x_2 &\geq 5 \\ \text{Minimize } -x_1 + x_2 & \end{aligned}$$

This is actually the initial no-finite-maximum problem of Chapter V, with all of the signs of the first two requirements equations reversed and with the above minimization functional replacing the maximization functional of Chapter V. This has no finite minimum, i.e., x_1, x_2, x_3, and x_5 can be increased indefinitely to satisfy the equations implicit in the above three inequalities, and, as this is done, x_1 always increases twice as much as does x_2. Consequently, $-x_1+x_2$ approaches $-\infty$.

The dual of this problem is:

$$\begin{aligned} 2y_1 - y_2 + y_3 &\leq -1 \\ -y_1 + 2y_2 + y_3 &\leq 1 \\ \text{Maximize } -2y_1 - 2y_2 + 5y_3 & \end{aligned}$$

There is no solution to this dual—finite, or otherwise.

Now suppose we combine this problem and its dual into a single game problem to get a skew-symmetric matrix, where $W_1=x_1z$, $W_2=x_2z$, $W_3=y_1z$, $W_4=y_2z$, $W_5=y_3z$, $W_6=z$, and

$$\overset{6}{\underset{j=1}{W_j}} \geq 0:$$

$$\begin{aligned} 0W_1 + 0W_2 + 2W_3 - 1W_4 + 1W_5 + 1W_6 &\leq 0 \\ 0W_1 + 0W_2 - 1W_3 + 2W_4 + 1W_5 - 1W_6 &\leq 0 \\ -2W_1 + 1W_2 + 0W_3 + 0W_4 + 0W_5 - 2W_6 &\leq 0 \\ 1W_1 - 2W_2 + 0W_3 + 0W_4 + 0W_5 - 2W_6 &\leq 0 \\ -1W_1 - 1W_2 + 0W_3 + 0W_4 + 0W_5 + 5W_6 &\leq 0 \\ -1W_1 + 1W_2 + 2W_3 + 2W_4 - 5W_5 + 0W_6 &\leq 0 \end{aligned}$$

Although the original minimization problem has no finite minimum and the dual has no feasible solution, since a game always has a solution, the above combined game problem has a solution. In fact, we can solve it as a linear programming problem by adding the specification that

$$\sum_{j=1}^{6} W_j = 1,$$

by adding an additional variable to each of the above inequalities, then subtracting the first (second, etc.) equation from the last equation, and minimizing the last above inequality (after first turning it into an equation):

$$\begin{array}{rcl}
1W_1 + 1W_2 + 1W_3 + 1W_4 + 1W_5 + 1W_6 & & = 1 \\
-1W_1 + 1W_2 + 0W_3 + 3W_4 - 6W_5 - 1W_6 - 1W_7 & + 1W_{12} & = 0 \\
-1W_1 + 1W_2 + 3W_3 + 0W_4 - 6W_5 + 1W_6 - 1W_8 & + 1W_{12} & = 0 \\
1W_1 + 0W_2 + 2W_3 + 2W_4 - 5W_5 + 2W_6 - 1W_9 & + 1W_{12} & = 0 \\
-2W_1 + 3W_2 + 2W_3 + 2W_4 - 5W_5 + 2W_6 - 1W_{10} & + 1W_{12} & = 0 \\
0W_1 + 2W_2 + 2W_3 + 2W_4 - 5W_5 - 5W_6 - 1W_{11} & + 1W_{12} & = 0 \\
\text{Minimize } -1W_1 + 1W_2 + 2W_3 + 2W_4 - 5W_5 + 1W_{12} & &
\end{array}$$

An optimum solution to this skew-symmetric game problem finally emerges as:

$$\begin{array}{r} -1\\ 1\\ 2\\ 0\\ 0\\ 0 \end{array}
\begin{bmatrix} W_1\\ W_2\\ W_4\\ W_9\\ W_{10}\\ W_{11} \end{bmatrix}
=
\underset{z_{00} = 0}{\begin{bmatrix} 1/2\\ 1/2\\ 0\\ 1/2\\ 1/2\\ 1 \end{bmatrix}}
-
\underset{z_{03} = 1}{\overset{2}{\begin{bmatrix} -3/6\\ 15/6\\ -1\\ -27/6\\ 27/6\\ 1 \end{bmatrix}}} W_3
-
\underset{z_{05} = 1}{\overset{-5}{\begin{bmatrix} 21/6\\ -15/6\\ 0\\ 51/6\\ 45/6\\ 0 \end{bmatrix}}} W_5$$

$$-
\underset{z_{06} = 2/6}{\overset{0}{\begin{bmatrix} 2/6\\ 8/6\\ -4/6\\ -3\\ 2/6\\ 38/6 \end{bmatrix}}} W_6
-
\underset{z_{07} = 4/6}{\overset{0}{\begin{bmatrix} 1/6\\ 1/6\\ -2/6\\ -3/6\\ -1/6\\ -2/6 \end{bmatrix}}} W_7
-
\underset{z_{08} = 2/6}{\overset{0}{\begin{bmatrix} 2/6\\ -4/6\\ 2/6\\ 1\\ -2\\ -4/6 \end{bmatrix}}} W_8
-
\underset{z_{0,12} = 0}{\overset{1}{\begin{bmatrix} -3/6\\ 3/6\\ 0\\ -9/6\\ 7/6\\ 0 \end{bmatrix}}} W_{12}$$

This means that B should play only strategies No. 1 and No. 2, and these should be given equal weights of 1/2. This is an optimum strategy, since the bottom row of z's tells us we have reached an optimum solution.

This provides us with an optimal strategy for B (and, of course, for A). But this is clearly no solution to the original linear programming problem, since we have $W_6=z=0$. This does not *necessarily* mean that there is *no finite minimum* for the *functional*, for there *might* be another optimal solution to this game with $z>0$. It does, however, mean that, even if the original linear programming problem and its dual do have optimal x's and y's (i.e., x's and y's which minimize and maximize their respective *functionals*), this set of optimal x's or y's is *unbounded*—for z will *never* equal zero, if both the optimal x's and y's are finite.[1] In this simple case, however, it is easily seen that there is no finite minimum for the original linear programming problem and no feasible solution for its dual, so there is no optimal strategy for the game which would result in a $z>0$.

Unbounded set of optimal x's or y's. Now consider a simple linear programming problem that leads to a game in which optimal strategies exist with both $z=0$ and $z>0$.[2] From the preceding paragraph, it should follow that this is indeed a problem having finite optimums for the *functionals* (for it is possible to find a $z>0$), but having the set of optimum x's or y's unbounded (for it is possible to find a $z=0$):

$$y_1 - y_2 \leq -1$$
$$\text{Maximize } y_1 - y_2$$

The dual of this problem is:

$$x_1 \geq 1$$
$$-x_1 \geq -1$$
$$\text{Minimize } -x_1$$

First looking at the minimization dual, it is clear that necessarily $x_1=1$, if both requirements inequalities are to be satisfied. For it is otherwise impossible, at the same time, to have $x_1 \geq 1$ and $-x_1 \geq -1$.

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[1] Cf. Luce and Raiffa, *Games and Decisions* (New York: John Wiley and Sons, 1957), pp. 420–423 for a rigorous mathematical proof. See also Dorfman, Samuelson, and Solow, *Linear Programming and Economic Analysis* (New York: McGraw-Hill Book Company, Inc., 1958), pp. 453–462.

[2] This problem is taken from Dorfman, Samuelson, and Solow, *op. cit.* A somewhat similar problem is also cited in S. Vajda, *The Theory of Games and Linear Programming* (London: Methuen and Co. Ltd., 1956), p. 77.

So there is a finite minimum (-1), and the "set" of x's is bounded $(x_1=1)$. It is also clear, in the maximization problem, that *all* pairs of values for y_1 and y_2 in which $y_2=y_1+1$ constitute optimal solutions to the maximization problem—subject, of course, to the usual non-negative-variables restriction. In brief, there *is* an optimal value for the maximization problem and its dual—namely, $Z=y_1-y_2=\not{z}=-x=-1$—but that same optimal value can result from an infinite number of pairs of values for y_1 and y_2.

Now that we have reasoned out the answer to this linear programming problem, consider what happens when we combine it and its dual into a game. Moving the right-side coefficients of the three requirements inequalities to the left side, then changing all signs of the two minimization inequalities (to change these inequalities from ≥ 0 to ≤ 0), subtracting the maximization functional from the minimization functional, and multiplying all terms of these four inequalities by z, we have:

$$\begin{aligned}
0W_1 + 1W_2 - 1W_3 + 1W_4 &\leq 0\\
-1W_1 + 0W_2 + 0W_3 + 1W_4 &\leq 0\\
1W_1 + 0W_2 + 0W_3 - 1W_4 &\leq 0\\
-1W_1 - 1W_2 + 1W_3 + 0W_4 &\leq 0
\end{aligned}$$

where $W_1=x_1z$, $W_2=y_1z$, $W_3=y_2z$, $W_4=z$, and

$$\overset{4}{\underset{j=1}{W_j}} \geq 0.$$

This game problem, too, we may solve by turning it back into a new linear programming problem. Following the now familiar procedure, we add one variable to each of the above four inequalities, make the last of these four resulting equations our new minimization functional, subtract the first (second, etc.) equation from the last, and place at the top the additional requirements equation of

$$\sum_{j=1}^{4} W_j = 1.$$

This results in:

$$\begin{aligned}
1W_1 + 1W_2 + 1W_3 + 1W_4 \qquad\qquad\qquad &= 1\\
-1W_1 - 2W_2 + 2W_3 - 1W_4 - 1W_5 + 1W_8 &= 0\\
0W_1 - 1W_2 + 1W_3 - 1W_4 - 1W_6 + 1W_8 &= 0\\
\text{Minimize } -1W_1 - 1W_2 + 1W_3 + 1W_8 = \not{z} - Z &= 0 \text{ (optimally)}
\end{aligned}$$

If we solve this system for W_1, W_2, and W_3, and W_6, we get:

$$\begin{array}{r} -1\\ -1\\ 1\\ 0 \end{array}\begin{bmatrix} W_1\\ W_2\\ W_3\\ W_6 \end{bmatrix} = \underset{z_{00}\,=\,0}{\begin{bmatrix} 0\\ 1/2\\ 1/2\\ 0 \end{bmatrix}} - \underset{z_{04}\,=\,0}{\overset{0}{\begin{bmatrix} -1\\ 3/2\\ 1/2\\ 0 \end{bmatrix}}} W_4 - \underset{z_{05}\,=\,1/3}{\overset{0}{\begin{bmatrix} -1/3\\ 1/2\\ -1/6\\ -2/3 \end{bmatrix}}} W_5$$

$$- \underset{z_{07}\,=\,1/3}{\overset{0}{\begin{bmatrix} 2/3\\ -1/2\\ -1/6\\ 1/3 \end{bmatrix}}} W_7 - \underset{z_{08}\,=\,1/3}{\overset{1}{\begin{bmatrix} -1/3\\ 0\\ 1/3\\ -2/3 \end{bmatrix}}} W_8$$

The z's at the bottom tell us that we have reached an optimum strategy for B for which the value of the game is zero, and we note that that strategy is for B to play strategies No. 2 and No. 3 with equal frequency. We also note that, with $W_4=0$, we do not have a solution to the original linear programming problem.

As we already know, however, although an infinite number of pairs of values for y_1 and y_2 exist such that $y_2=y_1+1$, the maximization functional (and hence the dual of its problem) does have an optimal feasible solution; so there must exist a solution in which $z>0$. Indeed, this is also revealed by the above bottom row of z's. Since the dual of a skew-symmetric matrix is the dual of itself, we can conclude that, for the dual, $z_{05}=z_{50}=z_{10}=1/3$, $z_{07}=z_{70}=z_{30}=1/3$, and $z_{08}=z_{80}=z_{40}=1/3$. But $z_{40}=W_4=z$. Therefore, an additional optimal solution to the game must be $W_1=1/3$, $W_2=0$, $W_3=1/3$, and $W_4=1/3$.

In fact, if we choose the hard way to verify this, we find that:

$$\begin{array}{r} -1\\ 1\\ 0\\ 0 \end{array}\begin{bmatrix} W_1\\ W_3\\ W_4\\ W_5 \end{bmatrix} = \underset{z_{00}\,=\,0}{\begin{bmatrix} 1/3\\ 1/3\\ 1/3\\ 0 \end{bmatrix}} - \underset{z_{02}\,=\,0}{\overset{-1}{\begin{bmatrix} 2/3\\ -1/3\\ 2/3\\ 0 \end{bmatrix}}} W_2 - \underset{z_{06}\,=\,1/2}{\overset{0}{\begin{bmatrix} 0\\ -1/2\\ 1/2\\ -3/2 \end{bmatrix}}} W_6$$

$$\begin{array}{cc} 0 & 1 \\ -\begin{bmatrix} 1/3 \\ -1/6 \\ -1/6 \\ -1/2 \end{bmatrix} W_7 & -\begin{bmatrix} -1/3 \\ 2/3 \\ -1/3 \\ 1 \end{bmatrix} W_8 \\ z_{07} = 1/2 & z_{08} = 0 \end{array}$$

Dividing W_1, W_2, W_3 by $W_4 = z$, we thus have an optimal solution to the linear programming problem and its dual of $x_1 = 1$, $y_1 = 0$, $y_2 = 1$.

Now any positive weighted average of the above two optimal game strategies for this game will also be an optimal strategy. Therefore, there is an infinite number of optimal game strategies and there is an infinite number of pairs of values for y_1 and y_2 which will satisfy the maximization functional. This follows because, in general, if we define our weights as $(1-\alpha)$ and α for our two strategies, and if we let

$$\overset{4}{\underset{j=1}{W_j'}} \left[\text{specifically, } \overset{4}{\underset{j=1}{W_j'}} = (0, 1/2, 1/2, 0)\right]$$

and

$$\overset{4}{\underset{j=1}{W_j''}} \left[\text{specifically, } \overset{4}{\underset{j=1}{W_j''}} = (1/3, 0, 1/3, 1/3)\right]$$

stand for the above two strategies, for the other infinite number of optimal game strategies we should have:

$$\begin{aligned}(W_1, W_2, W_3, W_4) &= [(1-\alpha)W_1' + \alpha W_1'', (1-\alpha)W_2' + \alpha W_2'', \\ &\qquad (1-\alpha)W_3' + \alpha W_3'', (1-\alpha)W_4' + \alpha W_4''] \\ &= \left(\frac{\alpha}{3}, \frac{1-\alpha}{2}, \frac{3-\alpha}{6}, \frac{\alpha}{3}\right),\end{aligned}$$

where $0 < \alpha < 1$.

For all solutions to our specific linear programming problem, then, we should have:

$$\begin{aligned}(x_1, y_1, y_2) &= \left(\frac{W_1}{W_4}, \frac{W_2}{W_4}, \frac{W_3}{W_4}\right) = \left(\frac{\alpha}{3}.\frac{3}{\alpha}, \frac{1-\alpha}{2}.\frac{3}{\alpha}, \frac{3-\alpha}{6}.\frac{3}{\alpha}\right) \\ &= \left(1, \frac{3-3\alpha}{2\alpha}, \frac{3-\alpha}{2\alpha}\right),\end{aligned}$$

where $0<\alpha<1$. Clearly, then, whatever value we choose for α (between 0 and 1), it will always be true that each of the infinite number of mixed strategies will result in $x_1=1$. Similarly, although the infinite number of different values for α will result in an infinite number of different values for y_1 and y_2, in all cases it will be true that $y_1-y_2=-1$, for

$$\frac{3-3\alpha}{2\alpha}-\frac{3-\alpha}{2\alpha}=\frac{-2\alpha}{2\alpha}=-1.$$

CHAPTER X

EPILOGUE: LINEAR PROGRAMMING, GAME THEORY, AND ECONOMICS

The principal objective of this book has been to impart an understanding of the largely mathematical rationale underlying the techniques of linear programming and game theory. Indeed, Chapters II–IX inclusive revolved almost entirely around that objective. Nevertheless, to put the rather abstract developments of those chapters in proper perspective, a concrete orientation problem was outlined in the latter part of Chapter I, and certain aspects of this same problem were also considered in Chapters II, VI, VIII, and IX.

It was further stated in Chapter I that the academician tends to view linear programming and game theory techniques as analytical tools, while the businessman tends to view these techniques as decision-making aids. Now in looking at our orientation problem, our view has been, at least implicitly, that of business management—for the focus of our attention has usually been upon discovering an optimal solution to the problem. Neither explicitly nor implicitly have we said anything about the (academic) economist's viewpoint.

An optimal solution to a problem is not the thing upon which the economic theorist primarily tends to focus his attention. Rather, the economist will generally focus his attention more upon the *characteristics* of that optimal solution, once such a solution has been found. For this reason, it is only appropriate to conclude this book with a brief consideration of linear programming and game theory techniques as analytical tools of the economist.

LINEAR PROGRAMMING AND ECONOMICS

Certainly the most complete and outstanding work on the relationship between linear programming and economic analysis is the substantial volume of Dorfman, Samuelson, and Solow.[1] Perhaps the most enlightening example (for our purposes) of the relationship between linear programming and economic analysis considered by these authors is concerned with general-equilibrium theory.[2] This is also one of the most complicated problems, and we must content ourselves with a cut version of only some of the main features. Such a version is no substitute for the full treatment, but it will suffice for our purposes and it should also encourage the economist to expose himself to the full treatment.[3]

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[1] *Linear Programming and Economic Analysis* (New York: McGraw-Hill Book Company, Inc., 1958).

[2] *Ibid.*, Chapter 13, especially pp. 351–375.

Walras-Cassel system. Consider a Walras-Cassel general-equilibrium system embracing an economy with m resources (or factors of production) and n final commodities.[4] For this system, we adopt the following symbolic terminology:

r_i = the amount of the ith resource supplied;
x_j = the amount of the jth commodity produced;
a_{ij} = the physical amount of the ith resource used in producing one unit of the jth commodity.

Note that each a_{ij} is an input coefficient of the Walras-Leontief type.

Under this symbolic terminology, then, producers of the first commodity (x_1) would *demand* $a_{11}x_1$ units of the first resource (r_1), and producers of the nth commodity (x_n) would demand $a_{1n}x_n$ units of this same first resource. Since r_1 is the *supply* of the first resource, if we equate demand and supply, we have:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = r_1$$

Since similar considerations apply for each resource,

$$\overset{m}{\underset{i=1}{r_i,}}$$

we have m equations:

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = r_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = r_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = r_m. \end{array} \tag{1}$$

Now if p_j is the *price* of the jth commodity (x_j) and v_i is the *price* of the services of the ith resource (r_i), we can say that the market-demand equations for the n commodities are:

$$\begin{array}{l} x_1 = F_1(p_1, \cdots, p_n; v_1, \cdots, v_m) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ x_n = F_n(p_1, \cdots, p_n; v_1, \cdots, v_m), \end{array} \tag{2}$$

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[3] For the convenience of such people, both the symbols and the numerical example cited in this section are precisely those of Dorfman, Samuelson, and Solow.

[4] This simplification can be (and ultimately is) amended to include intermediate goods by Dorfman, Samuelson, and Solow.

the v_i factor prices appearing in these demand functions to allow for changes in demand due to shifts in the level and distribution of income.

A condition of long-run competitive equilibrium states, of course, that the price of each commodity must equal its unit costs. Moreover, with intermediate goods washed out, unit costs necessarily consist of payments for resources. Consequently, each unit of the total x_1 units of the first commodity requires a_{11} units of the first resource (r_1), a_{21} units of the second resource (r_2), and so on to a_{m1} units of the mth resource (r_m). For the first commodity, then, we have, in equilibrium, that:

$$a_{11}v_1 + a_{21}v_2 + \cdots + a_{m1}v_m = p_1.$$

Since there are n commodities, we thus have a system of n equations:

$$\begin{aligned} a_{11}v_1 + a_{21}v_2 + \cdots + a_{m1}v_m &= p_1 \\ a_{12}v_2 + a_{22}v_2 + \cdots + a_{m2}v_m &= p_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{1n}v_1 + a_{2n}v_2 + \cdots + a_{mn}v_m &= p_n. \end{aligned} \tag{3}$$

Thus the matrix of the a_{ij} coefficients for equations (3) is the transpose of the matrix of coefficients for equations (1), which gives some hint of what is coming.

Just as one can derive the zero-degree homogeneous *demand* functions (2) for *commodities* as functions of the p_j and v_i, one can also derive zero-degree homogeneous *supply* functions for *resources* as functions of the p_j and v_i. In our case, however, we shall simply take the factor supplies (r_i) as given.[5]

Looking at our system of the three sets of equations (1), (2), and (3), we must ask ourselves whether this system necessarily has a solution which makes economic sense. Equations (1) consist of m equations in n unknowns. If $m > n$, these equations will certainly have no solution, unless the a_{ij} coefficients and the resource supplies stand in a very special relationship to each other, and there is no reason to expect such a happy circumstance.

Even if $m = n$,[6] there is still no assurance that our system has an economically meaningful solution. To see this, consider a system with two final goods, two resources, input coefficients of $a_{11}=1$, $a_{12}=2$, $a_{21}=3$, $a_{22}=4$, and given factor supplies of $r_1=2$, $r_2=20$. This turns the general equations (1) into the specific equations of:

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[5] Here, too, Dorfman, Samuelson, and Solow ultimately expand their system to include supply functions for resources.

[6] This was the case in the Walrasian system, i.e., Walras "disposed" of the solution-existence problem by showing that the number of equations was equal to the number of unknowns in his system.

$$(1)' \qquad \begin{aligned} x_1 + 2x_2 &= 2 \\ 3x_1 + 4x_2 &= 20 \end{aligned}$$

Suppose, further, that the elasticity of demand for both of our final goods is (independent of factor prices) unity, so that $p_1x_1=8$ and $p_2x_2=10$. Then the specific counterpart of the general equations (2) is:

$$(2)' \qquad \begin{aligned} x_1 &= \frac{8}{p_1} \\ x_2 &= \frac{10}{p_2} \end{aligned}$$

And, of course, the specific counterpart of the general equations (3) is:

$$(3)' \qquad \begin{aligned} v_1 + 3v_2 &= p_1 \\ 2v_1 + 4v_2 &= p_2 \end{aligned}$$

At this point, we really need not concern ourselves with (2)′ and (3)′. Equations (1)′ immediately tell us that there is no solution to this system which makes economic sense. Our first commodity uses the two resources in a ratio of 1:3 and the second commodity uses them in a ratio of 2:4. Since the supply of the two factors stands in a ratio of 1:10, however, how can $r_2(=20)$ be fully employed?

Indeed, if we solve equations (1)′, we immediately encounter precisely the same sort of snag as we struck in our first solution to the blending problem equations of Chapter I: namely, a negative variable. For the solution to equations (1)′ results in $x_1=16$, $x_2=-7$, and this has no real economic meaning.

If the economic system of equations (1), (2), and (3) does not represent a system to which a solution necessarily exists, can that system be amended in such a way as to assure us that a meaningful solution always exists? The answer to this question is an affirmative one, and the amending process leads us into the equivalent of a linear programming problem and its dual.

The amended system. One of the jobs of the general-equilibrium system is to decide, in terms of the given data, which resources shall be free and which shall have a positive price. Consequently, the list of resources

$$\left(\overset{m}{\underset{i=1}{r_i}}\right)$$

of the Walras-Cassel model should be all-inclusive. But equations (1) say, in effect, that there are no free resources, i.e., that all resources are fully employed. Diagrammatically speaking, however, there is no reason why the (assumed-to-be) perfectly inelastic supply curve (a line perpendicular to the x-axis) of say r_k should not hit the x-axis to the right of the point at which the demand curve for r_k hits the x-axis. Such a situation would mean a zero price for r_k and something less than full employment of that resource.

An obvious way to incorporate this possibility into equations (1) is to supplant the = signs of (1) with $\leq$ signs. Our first set of equations then becomes:

$$\text{(1a)} \qquad \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq r_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq r_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq r_m, \end{array}$$

and we understand that the economic meaning is that, if the strict inequality holds in any row of (1a), the resource of that row (say r_2) is less than fully employed and r_2's price, v_2, is zero.

A somewhat analogous adjustment is also indicated for equations (3), namely, the unit-costs-equal-price equations. The rationale behind these equalities in equilibrium is, of course, that: (a) if price exceeds unit costs, both factor prices and output will rise, thus lowering price and increasing costs; and (b) if unit costs exceed price, the reverse process will take place. There is, however, a limit to the reverse process, since output cannot fall below zero. Consequently, it is perfectly feasible for unit costs to exceed price at zero output (in the words of Dorfman, Samuelson, and Solow, "It is no accident that silk diapers are so rare").

An obvious way to incorporate this phenomenon into equations (3) is to supplant the = signs of (3) with $\geq$ signs. The third set of equations then becomes:

$$\text{(3a)} \qquad \begin{array}{l} a_{11}v_1 + a_{21}v_2 + \cdots + a_{m1}v_m \geq p_1 \\ a_{12}v_1 + a_{22}v_2 + \cdots + a_{m2}v_m \geq p_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{1n}v_1 + a_{2n}v_2 + \cdots + a_{mn}v_m \geq p_n, \end{array}$$

and we understand that the economic meaning is that, if the strict inequality holds in any row of (3a), the output of the commodity represented by that row (say x_2 for the second row) is zero.

Now let us return to the specific sets of equations (1)′, (2)′, and (3)′, amending (1)′ and (3)′ to get:

1a)′

$$x_1 + 2x_2 \leq 2$$
$$3x_1 + 4x_2 \leq 20$$

(2)′

$$x_1 = \frac{8}{p_1}$$
$$x_2 = \frac{10}{p_2}$$

(3a)′

$$v_1 + 3v_2 \geq p_1$$
$$2v_1 + 4v_2 \geq p_2$$

Suspecting that $3x_1+4x_2<20$ (i.e., a strict inequality holds) and that $v_2=0$, since we then have from (3a)′ that $v_1=p_1$ and $2v_1=p_2$, it follows that $p_2=2p_1$.

We can now substitute $2p_1$ for p_2 in the second equation of (2′) to get:

(2)″

$$p_1 = \frac{8}{x_1}$$
$$p_1 = \frac{5}{x_2}$$

From this we deduce that, since

$$\frac{8}{x_1} = \frac{5}{x_2}, \ 5x_1 - 8x_2 = 0.$$

If we combine this last equation with the first equation of (1a)′, and solve these, we get $x_1=8/9$, $x_2=5/9$.

From (2)′ it then follows that $p_1=9$, $p_2=18$. And from both equations of (3a)′ it follows that $v_1=9$. Moreover, the first resource is fully employed, and from the left side of the second inequation of (1a)′ we can see that $3\times 8/9+4\times 5/9=44/9$, which is the amount of the second resource used, leaving 136/9 of r_2 unemployed.

This is the sensible and quicker way to solve this problem. We could, however, have directly unlimbered our linear programming techniques on this problem. For, if we examine (1a) and (3a), it looks suspiciously as though we were looking at the constraints of dual linear programming problems. Indeed, we might even specify the functional for (1a) as Maximize $p_1x_1 + \cdots + p_nx_n$ (= value of output), and for (3a) as Minimize $r_1v_1 + \cdots + r_mv_m$ (= total factor income), in both cases for non-negative variables.

At this point, we should note two ways in which this problem is different from any of the problems we have solved in the preceding chapters. In the first place, while we know that none of the a_{ij}, r_i, v_i, p_j, or x_j can be negative, inequations (1a) and (3a) are otherwise in a completely general form, i.e., we have no *specific* values for the coefficients and the variables. Since we also know (from Chapter V) that not every linear programming problem necessarily has a feasible solution, how do we know that this particular generalized one always has? Secondly, inequations (1a) and (2a) have, instead of the conventional *two* unknowns, *three* unknowns: namely, v_i, p_j, and x_j (remember, we took the r_i as given).

Turning to the first of these two points, the two linear programming problems do necessarily have feasible solutions—thanks to the fact that there can be neither negative variables nor negative a_{ij} coefficients. In other words, no matter what the values of the right sides of inequations (1a) and (3a), we can always find values small enough for the x_j and large enough for the v_i to make both sets of inequations valid. Knowing that solutions to the linear programming problems do exist, we also know (from the fundamental duality theorem) that

$$\sum_{=1}^{n} p_jx_j \text{ and } \sum_{i=1}^{m} r_iv_i$$

are equal: namely, that the maximization functional (total expenditure) equals the minimization functional (total income), *provided* an equilibrium solution to the general equilibrium problem exists (which we have yet to prove) which is also a solution to the linear programming problems. Consequently, although we can so far be sure of little else, it should now seem clear that every competitive equilibrium system implicitly contains a maximization-of-value-output problem and a minimization-of-factor-returns problem.

While we know that the dual linear programming problems have solutions, we do not yet know that an equilibrium solution exists to the general equilibrium problem. The reason for our ignorance on this point is associated with the above-mentioned fact that the linear programming problems involve the *three* unknowns of n p's, n x's, and m v's. The n p's are on the right side of inequations (3a), and we were able to deduce that solutions existed to the linear programming problems, partially because *any* set of non-negative p's could result in a solution to the linear programming problems.

It is not true, however, that *any* set of non-negative p's will assure the existence of an equilibrium solution to the general equilibrium system. The linear programming problem, viewed purely as such, is concerned only with inequations (1a) and (3a). The general equilibrium system, however, also includes equations (2). We can choose any set of p's and solve the linear programming problems to get the acceptable sets of x's and v's for the linear programming problems. But there is clearly no assurance that these sets of p's, x's and v's will also satisfy equations (2). The basic question is, then, whether it is a certainty that there is at least one set of p's which will yield sets of x's and v's that will also satisfy equations (2).

For our purposes, it is perhaps necessary to go no further than to accept on faith the fact that Kakutani's fixed-point theorem assures us that the answer to this last question is in the affirmative.[7] It can then be shown that there does indeed *exist* an equilibrium solution to the general equilibrium system, and this solution is also *unique* in p and x, if we accept the axiom of revealed preference.

From this discussion of the amended system, then, we have garnered two main points of information: (1) every competitive equilibrium system implicitly contains a maximization-of-value-output problem and a minimization-of-factor-returns problem; (2) not only do solutions to the linear programming problems exist, but there also exists a unique solution to the general equilibrium system. It is a fair question, from the economist's viewpoint, whether proving the mere existence of an equilibrium solution to the general equilibrium system is really worth all this bother.

Importance of existence theorems. As one well-known economist has accurately said, " . . . the existence theorems . . . are mainly reassuring. It is nice to know that someone somewhere has shown that our model does indeed have a solution."[8] But if this is an accurate statement in the literal sense, the flavor it seems to carry appears to be somewhat less accurate. There is something more positive than this to be said for proof of the existence of an equilibrium solution to the general equilibrium system,[9] and Dorfman, Samuelson, and Solow have said it about as well as anyone could ask.[10]

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[7] Cf. Dorfman, Samuelson, and Solow, pp. 371–374.

[8] W. J. Baumol, "Activity Analysis in One Lesson," *American Economic Review*, December, 1958, p. 872. It might be added that this is a good article, even if one cannot really master activity analysis in one lesson.

There can be no doubt that whatever the true model of the real economic world may look like, a solution to it would have to exist, since the real economic world does function. But it is less than certain that this real-economic-world solution is one involving a perfectly competitive equilibrium. In any event, we cannot be sure that the properties of our abstract model are an accurate portrayal of the real world.

Now the purpose of the economist's abstract model is to enhance his study of the real economic world. For this reason, it is important to know whether the collection of supply-demand relations of the model gives an accurate portrayal of what is important in the real economic world. Proof of the existence of a solution to the abstract model is not a *sufficient* condition for guaranteeing that such is the case, but it is a *necessary* condition. Thus, for example, while we cannot know that the amended system is an accurate portrayal of the real economic world, at least we do know that it satisfies one necessary condition for such a portrayal that the initial Walras-Cassel model does not satisfy. This would seem to be more than a minor contribution, and the amended system *does* make more economic sense.

What sort of conclusion should we draw, then, about the importance of linear programming techniques to economic analysis? There can scarcely be any question about the value of these techniques to applied problem solving. Hence, there can be little doubt about their importance to either business management as a decision-making aid or to the economist concerned with certain types of specific questions.

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[9] While analogies can be misleading, analogies to this particular case of proving the existence of a solution to a problem are to be found in the world of pure mathematics. Mathematicians spent a lot of time, for example, during the seventeenth and eighteenth centuries, trying to find general equations for solving equations of the fifth degree and higher by radicals. But not until 1799 did Gauss develop the first complete proof that such equations could, in fact, be solved at all. And it required a few more years for Abel and Ruffini (independently) to conceive the revolutionary idea of proving the *impossibility* of the approach *by radicals*. It is probably conservative to say that most mathematicians regard these as worthwhile contributions.

[10] *Ibid.*, pp. 350–351.

The importance of these techniques, to the economist concerned with such general problems as the equilibrium one we have considered, may be currently more debatable, at least in the sense that we cannot yet be certain how vital the role so far played by these techniques will ultimately prove to have been; but if this is so, it is equally true that we cannot yet lightly conclude that that role will not prove to have been a crucial one.

GAME THEORY AND ECONOMIC ANALYSIS

There is little disagreement as to the importance of game theory techniques to economic analysis. These techniques have some indirect value in those cases where there is something to be gained by turning a linear programming problem into a game for solution purposes. Beyond this there is, at least currently, not too much to be said for game theory techniques as an economic tool, and for fairly obvious reasons.

The theory outlined in Chapter VII, and the games solved in Chapters VIII and IX, were zero-sum (i.e., *A*'s gains were equal to *B*'s losses) two-person (namely, just *A* and *B*) games. Unfortunately, few problems in economics take the form of zero-sum two-person games. Rather, such problems tend to run to *non*-zero-sum two-person games—the classic example being a bargaining "game" between trade union and management, in which both players may conceivably either gain or lose; or such problems tend to run to more-than-two-person games—oligopoly being an obvious case. But the current state of game theory in these more complicated games is far from satisfactory.

For essentially the same reasons, it may also be said that game theory techniques do not have a great deal more to contribute to business-management decisions than they have to economic analysis. One possible (and perhaps potentially important) exception to this is such unconventional "games" as business management must play with "Nature" as its opponent, for Nature will not behave like the typical malevolent opponent.[11] Given such a game matrix, there will frequently be probability coefficients for *Nature's* strategies which make it a matter of *indifference* to management which one of its own strategies it follows. If management is in a position to make an intelligent *estimate* of what the *true* probabilities for each of Nature's strategies are, then a comparison of these estimated true probabilities to the indifference probabilities will reveal an estimate of both management's best strategy and the degree of that strategy's "bestness."

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[11] E. G. Bennion, "Capital Budgeting and Game Theory," *Harvard Business Review*, November–December, 1956.

CONCLUSION

Does the preceding assessment mean that linear programming and game theory techniques are not important enough to economics to warrant an economist's making the considerable investment in time required for a useful understanding of these techniques? It is certainly true that "Programming is mathematics, not economics."[12] But this is, of course, no argument for ignoring programming and game theory techniques. Programming is no more business than it is economics, but it does have much to offer business. The calculus is also mathematics, not economics—but the calculus is an indisputably powerful tool for the economic theorist. The real question is not, whether these various tools are economics (which they clearly are not), but rather whether they are genuinely useful tools to particular professions.

By this criterion, it can be categorically stated that linear programming techniques have a real contribution to make to business management. On the other hand, the economist who ignores programming techniques may not, in any important sense, fall as far behind his "competitors" as a business management almost surely will, although this is by no means a certainty. At a minimum, the economist owes it to himself to give at least a cursory examination to the array of economic problems in which these techniques can be used,[13] before deciding against the potential of such tools.

Even in the case of game theory, it is not an open-and-shut case that game theory techniques can be safely ignored.[14] Abstracting from the question of future improvements in the theory underlying non-zero-sum and more-than-two-players games (which is a questionable abstraction), this is still true.

From the viewpoint of business management, it is one thing to say that the types of games confronting it are decidedly limited in number, and it is quite another thing to say that the number of problems falling within these limited types is quantitatively unimportant. A little ingenuity, coupled with an understanding of game theory techniques, may well open substantial new horizons.

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[12] W. J. Baumol, *idem.*

[13] Merely thumbing through Dorfman, Samuelson, and Solow, *op. cit.*, without worrying about the mathematical arguments developed, should be genuinely helpful in this respect.

[14] A careful study of the recently published book by Martin Shubik, *Strategy and Market Structure* (New York: John Wiley and Sons, Inc.), 1959, might well show this to be a much too conservative statement.

From the viewpoint of both business management and the economist, it is not entirely irrelevant that close relationships exist between programming and game theory techniques. Practically speaking, the importance of this can be exaggerated. On the other hand, the process of acquiring an understanding of programming techniques takes one so far along the road towards understanding much of zero-sum two-person game theory that it should almost be worth one's time to go the rest of the way, merely to satisfy one's intellectual curiosity, if for no other reason.

Finally, from the economist's viewpoint, it should be recognized that game theory techniques, whatever their current shortcomings as a quantitative tool for coping with the complex problems of economics, may well have usefulness as a qualitative tool. Here, too, Dorfman, Samuelson, and Solow[15] cite a pertinent illustration in "the concept of coalitions maintained by means of side payments among the members" of opposing interest as a theoretical counterpart to cartels, adding that, had this vocabulary been in use in the day of Edgeworth and von Stackelberg, they would probably not have overlooked the possibility and importance of such coalitions.

Such is the main outline of the pros and cons, as the writer sees it, with respect to linear programming and game theory techniques. Each person must decide for himself whether, for his purposes, the balance is a debit or credit one. That the writer has assessed the balance as a large credit one is fairly obvious—for, if this were not the case, this book would never have been written.

[15] Dorfman, Samuelson, and Solow, *op. cit.*, p. 445.

BIBLIOGRAPHY

Allen, R. G. D. *Mathematical Economics.* London: Macmillan and Company, Ltd., 1956, Chapter 15 (Elementary Theory of Games) and 16 (Linear Programming).

Baumol, W. J. "Activity Analysis in One Lesson," *American Economic Review*, December, 1958.

Bennion, E. G. "Capital Budgeting and Game Theory," *Harvard Business Review*, Volume 34, Number 6.

Charnes, A., W. W. Cooper, and A. Henderson. *An Introduction to Linear Programming.* New York: John Wiley and Sons, Inc., 1953.

Dorfman, Robert, Paul A. Samuelson, and Robert M. Solow. *Linear Programming and Economic Analysis.* New York: McGraw-Hill Book Company, Inc., 1958.

Koopmans, T. C. (ed.). *Activity Analysis of Production and Allocation.* New York: John Wiley and Sons, Inc., 1951.

Luce, R. Duncan and Howard Raiffa. *Games and Decisions: Introduction and Critical Survey.* New York: John Wiley and Sons, Inc., 1957.

McKinsey, J. C. C. *Introduction to the Theory of Games.* New York: McGraw-Hill Book Company, Inc., 1952.

Neumann, John von and Oskar Morgenstern. *Theory of Games and Economic Behavior.* Princeton, New Jersey: Princeton University Press, 1944.

Symonds, Gifford H. *Linear Programming: The Solution of Refinery Problems.* New York: Esso Standard Oil Company, 1955.

Vajda, S. *Theory of Games and Linear Programming.* London: Methuen and Company, Ltd., 1956.

Wagner, Harvey M. "Advances in Game Theory," *American Economic Review*, June, 1958.

Wagner, Harvey M. "The Simplex Method for Beginners," *Operations Research*, Volume 6, Number 2.

Williams, J. D. *The Compleat Strategyst.* New York: McGraw-Hill Book Company, Inc., 1954.

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